

## Math appendix

Important numbers, important formulas, important rules, important functions and what they look like graphically.

### Numbers worth memorizing:

$$\pi = 3.14159\dots$$

$$e = 2.7183\dots$$

We often **normalize** a function or distribution in order to compare it with others. This is shown in many of the graphs below. Normalization is accomplished by dividing both the x-values and the y-values by constants, and plotting the resulting (now non-dimensional) values on the new axes. This process is something of an art, which comes in the choice of what scales to use in normalizing. An obvious choice for the constant is the maximum value. Upon dividing by the maximum, the resulting ratios must lie between 0 and 1. Often, however, there is no clear maximum value (especially on the x-axis), and we must choose something that is characteristic of the problem. Examples include the standard deviation of a distribution, or the period of the oscillation, or the length scale over which the value changes by factor of 2 or of  $e$ . (See graphs of exponentials.) The result is always a graph that has values that go from 0 to 1 or from 0 to a few. It also allows us to compare the shapes of functions to one another, as this normalization removes the role of the scales themselves. Note for example how all of the Gaussian curves plot on top of one another when normalized.

### Important functions

1. *Straight lines*.  $y = mx + b$ . (Figure 1) Here the slope on the plot is  $m$ , while the y-intercept is  $b$ . Lines are everywhere in geomorphology. They define the straight slopes of landslide-prone hillslopes. They relate the flux of regolith and slope angle for rainsplash and frost-creep processes.

2. *Negative exponentials* (Figure 2) are encountered in radioactive decay, in production profiles of cosmogenic nuclides...  $y = Ae^{-x/x^*}$ . These are characterized by two constants,  $A$  and  $x^*$ . The first is the maximum value, found at  $x=0$ . The second is the scale over which the function falls by a factor of  $e$ . We call this scale the e-folding scale. It is found graphically by finding the place at which the value of the function is  $1/e$  of  $A$ , or  $A/e$ . Recalling that  $e$  is about 3 (actually 2.7183...), this is roughly a third of the value of  $A$ , which is easy to estimate on the graph. These functions are encountered in the decay of radioactive nuclides.

3. *Positive exponentials* (Figure 3) are found in the unchecked growth of populations:  $y = Ae^{x/x^*}$ . In this case,  $x$  would stand for time. Exponential growth is what we expect in a population that grows at a rate dictated by the number of individuals in the population at any time. Just as negative exponentials, it too is characterized by two constants,  $A$  and  $x^*$ . The first is again the initial value, found at  $x=0$ . The second is the scale over which the function changes (this time *increases*) by a factor of  $e$  (the e-folding scale).

4. Closely related to exponentials is a function that approaches an *asymptote* as an exponential (Figure 4).  $y = A(1 - e^{-x/x^*})$ . Here the function is defined by the value of the asymptote,  $A$ , and by the rate at which the asymptote is approached. Again, this is set by a scale, we use  $x^*$ . Note the values of the function at three places: at  $x=0$ ,  $e(0) = 1$ , implying  $y = 0$ . At  $x = \infty$ ,  $e^{(-\infty)} = 0$ , implying  $y = A$ . Finally, at  $x=x^*$ ,  $y=A(1-e^{-1}) = A(1-(1/e)) = 0.63A$ . This approach toward an asymptote is found in systems in which both growth and decay occur, the asymptote reflecting a balance of growth and decay (also called secular equilibrium in radioactive decay series).

5. *Power law functions* (Figure 5) are very common in geomorphology.  $y = Ax^p$  They arise in drainage basin characteristics... They have the important property that they become straight lines when plotted on log-log graphs. The slope of the line is the power. You can see this by logging both sides of the equation:  $\log(y) = \log(A) + p\log(x)$ . This has the form of  $y = b + mx$  which we all recognize as a straight line with intercept  $b$  and slope  $m$ . This means that an easy way to evaluate the power in a power-law function is by plotting it in this manner. Power law functions are everywhere in geomorphology. To name two examples, power laws describe the relationship between the number of streams of one order with respect to the number in the next order (one of Horton's laws), and the relationship between slope and drainage area in a bedrock stream profile.

6. The *parabola* (Figure 6) is of course a special case of a power law.  $y = y_o + A(x - x_o)^2$ . But it is so commonly seen in geomorphology it is worth breaking out separately. Sand grains splashed up by raindrops carry out parabolic trajectories. Steady state hilltops have parabolic topographic profiles. Flow of a viscous fluid between two plates (like magma in a dike) has a parabolic velocity profile. The general form here accommodates a parabola centered not on  $[0,0]$  but on  $[x_o, y_o]$ . This can be seen by setting  $x = x_o$ . The value of  $y = y_o$ .

7. *Logarithmic functions* (Figure 7).  $y=A \log(x/x_o)$ . These are encountered in fluid mechanics. For example, the flow speed increases as a logarithm of height above the bed in an open channel flow. These functions have the property of rapidly increasing at first and much more slowly increasing thereafter.

8. Plots of two major *trigonometric functions* (Figure 8) are shown in order to emphasize the role of scaling in both the  $x$  and  $y$  dimensions. Recall that  $\sin(0) = 0$  and that  $\cos(0) = 1$ . Quite complicated looking graphs can be constructed by combining sin and cosine curves – this is the essence of the Fourier transform. The surface temperature of the earth carries out sinusoidal swings on both daily and annual time scales.

9. The *hyperbolic tangent* (Figure 9). We include this function because it serves to step smoothly from one value (here  $b-a$ ) to another ( $b+a$ ) over a specified distance, scaled by  $x^*$  and centered at  $x_o$ . The formula is  $y = b + a \tanh((x-x_o)/x^*)$ .

10. *Hyperbolic sine* and *hyperbolic cosine* (Figure 10) can also be scaled as shown on their plots. One encounters these functions in solutions for the displacement profile around a fault.

11. The *Gaussian* (Figure 11) is often encountered in error analysis, as errors are supposed to be normally, or Gaussianly distributed. The function is named for Karl Freidrich Gauss, a great

German mathematician and astronomer living 1777-1855.  $y = Ae^{-\left(\frac{x-x_o}{x^*}\right)^2}$  This is the classic bell-shaped curve. As written, it is centered on  $x=x_o$ . The value of  $x^*$ , (also known as the standard deviation if this is a probability density function (see below)), sets how sharply the curve falls off away from the peak value of  $A$ .

## Basic rules of thumb for manipulation of expressions

### Logs, powers and exponentials

laws of exponents

$$a^m a^n = a^{(m+n)}$$

$$\frac{a^m}{a^n} = a^{(m-n)}$$

$$a^0 = 1$$

$$a^{p/q} = \sqrt[q]{a^p}$$

$$a^{-t} = \frac{1}{a^t}$$

logarithms

Definition: If  $y = \log_a(x)$ , then  $a^y = x$ , where  $a$  is called the base of the logarithm.

Most important bases for us are 10, 2 and  $e$ .

Terminology:  $\log_e(x) = \ln(x)$  or the log base  $e$ , is also called the *natural logarithm*

Conversion from  $\log_{10}$  to  $\ln$ :  $\log_{10}(a) = \ln(a)/\ln(10) = \ln(a)/2.303$

laws of logarithms

$$\log(xy) = \log(x) + \log(y)$$

$$\log(x/y) = \log(x) - \log(y)$$

$$\log(x^b) = b \log(x)$$

### Trigonometry

In (Figure 12) we show a right triangle, one with a perpendicular ( $90^\circ$  or  $\pi/2$  radians) angle. The definitions of  $\sin$ ,  $\cos$  and  $\tan$  are as follows:

$$\sin = \text{opp/hyp}$$

$$\cos = \text{adj/hyp}$$

$$\tan = \text{opp/adj} = \sin/\cos$$

$$\sin^{-1}(x) = \text{invsin}(x) = \text{inverse sin}(x)$$

$$\text{if } \sin^{-1}(x) = a, \text{ then } \sin(a) = x$$

$$1 \text{ radian} = 360/2\pi = 57.3^\circ$$

*small angle approximation*:  $\sin(\alpha) = \tan(\alpha) = \alpha$  for very small angles,  $\alpha$  (where  $\alpha$  is taken to be in radians!). For the same reasons,  $\cos(\alpha) = 1$  for very small angles.

*Pythagorean theorem*. For a right triangle with hypotenuse  $h$  and sides  $a$  and  $b$ ,

$h^2 = a^2 + b^2$ . This can be manipulated to solve for the length of any unknown side given the other two. This is named for Pythagoras, the great Greek mathematician and philosopher who died around 497 B.C.

*angle formulas*

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\text{Law of sines: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

## Geometry

*Volume, area, and circumference*

$$\text{Circle: } C = 2\pi R = \pi D$$

$$A = \pi R^2 = \frac{1}{4} \pi D^2$$

$$\text{Sphere: } A = 4\pi R^2 = \pi D^2$$

$$V = \frac{4}{3} \pi R^3 = \frac{1}{6} \pi D^3$$

## Algebra

The **quadratic formula**. For an equation that may be written

$$ax^2 + bx + c = 0,$$

the roots (the values of x where the equation is 0) may be found using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

## Calculus

**Derivative**. The *first derivative* is the local slope of a function, say y(x). It is defined as the

$$\frac{dy}{dx} \equiv \lim_{dx \rightarrow 0} \frac{y(x + dx) - y(x)}{dx}$$

so that as the interval over which the slope is being evaluated shrinks, the local slope is better and better approximated (see **Figure 13**). The derivative is variously expressed as  $dy/dx$ ,  $y'(x)$ ,

$y'$ , even  $y_x$ . When you see the notation  $\frac{\partial z}{\partial x}$ , it signifies a *partial derivative*, and is read the partial

derivative of z with respect to x. This occurs when there are several variables involved, for example, when z is a function of x and y,  $z(x,y)$ , and means the derivative of z taken with respect to the variable x, while holding all other variables constant. The dimensions of a derivative are those of y over those of x.

Second derivative of a function,  $\frac{d^2y}{dx^2}$  is the curvature of the function, or the slope of the slope. It is positive if the slope is increasing with distance,  $x$ , and negative if the slope is declining with distance (Figure 14). Comparable notation for second derivative is  $d^2y/dx^2$ ,  $y''(x)$ ,  $y''$ , and  $y_{xx}$ . The dimensions of a second derivative are those of  $y$  over those of  $x$  squared.

While massive tables exist of both derivatives and integrals (e.g., in the front of any CRC Handbook of Chemistry and Physics), we list important examples here for convenience.

In the formulas to follow,  $a$  is a real constant,  $x$  is a variable,  $u$  is a function of  $x$ ,  $u(x)$ , and all trigonometric functions are measured in radians.

### Derivatives

1.  $\frac{d}{dx}(a) = 0$
2.  $\frac{d}{dx}(x) = 1$
3.  $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$
4.  $\frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx}$  -- the product rule
5.  $\frac{d}{dx} \ln(u) = \frac{1}{u} \frac{du}{dx}$
6.  $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$
7.  $\frac{d}{dx}(\sin(u)) = \cos(u) \frac{du}{dx}$
8.  $\frac{d}{dx}(\cos(u)) = -\sin(u) \frac{du}{dx}$
9.  $\frac{d}{dx}(\tan(u)) = \sec^2(u) \frac{du}{dx}$

The **integral** of a function is the area under the function, lying between the function and the  $x$ -axis (see Figure 15). A *definite integral* is taken over a specified interval,  $[a,b]$ , while for an *indefinite integral* this interval is not specified.

### Integrals

Indefinite integrals. To each of the solutions shown, you must add a “constant of integration”.

1.  $\int a dx = ax$
2.  $\int x^n dx = \frac{x^{n+1}}{n+1}$  except when  $n = -1$
3.  $\int \frac{1}{x} dx = \log(x)$

4.  $\int e^x dx = e^x$
5.  $\int e^{ax} dx = \frac{e^{ax}}{a}$
6.  $\int \log(x) dx = x \log(x) - x$
7.  $\int \sin(ax) dx = -\frac{1}{a} \cos(ax)$
8.  $\int \cos(ax) dx = \frac{1}{a} \sin(ax)$

A couple useful definite integrals:

1.  $\int_0^{\infty} e^{-ax} dx = \frac{1}{a}, a > 0$
2.  $\int_0^{\infty} e^{-a^2 x^2} dx = \frac{1}{2a} \sqrt{\pi}, a > 0$

The **mean value theorem** is used to assess formally the mean value (or average) of a function over a specified interval. This is shown graphically in **Figure 16**, and mathematically as:

$$\bar{y} = \frac{1}{b-a} \int_a^b y(x) dx$$

Note that if we rearrange this by multiplying by (b-a), it becomes

$$\bar{y}(b-a) = \int_a^b y(x) dx$$

The left hand side corresponds to the area of the box  $\bar{y}$  tall and  $(b-a)$  wide, while the right hand side corresponds to the area under the curve,  $y(x)$ . The mean value of the function is therefore formally the value of  $y$  at which these areas are equal. We make use of this, for example, in evaluating the mean flow velocity, given a flow velocity profile.

## Ordinary differential equations, ODEs

These contain a variable and its derivatives; they are limited to one variable – if more, then partial differential equations, PDEs.

They are classified according to several criteria:

- the highest degree of derivative in the equation (called the order of the equation)
- whether they contain variables as multiples or powers, or products of variables with derivatives (linear if not, nonlinear if so)
- whether coefficients are constants or not

examples:

$$\frac{dN}{dt} + aN = 0 \text{ first order, linear homogeneous ODE}$$

$$N = N_0 e^{-at}$$

We encounter this equation in the decay of radionuclides such as  $^{14}\text{C}$  used to date geomorphic surfaces.

$$\frac{dN}{dt} + aN = P \quad \text{first order, linear, nonhomogeneous ODE}$$

$$N = \frac{P}{a}(1 - e^{-at})$$

This describes a population with both new production or growth, represented by  $P$ , and decay or death, represented by  $aN$ . The solution is a function that approaches an asymptote, called secular equilibrium, in which growth and decay are perfectly balanced.

$$\frac{d^2z}{dx^2} = A \quad \text{second order, linear, nonhomogeneous ODE}$$

$$z = \frac{1}{2}Ax^2 + c_1x + c_2$$

where  $c_1$  and  $c_2$  are constants of integration, for which one must appeal to boundary conditions. In this example, if  $z$  is topographic elevation, the equation would represent a system in which topographic curvature is uniform (at  $A$ ). This occurs on steady hilltops, and leads to a parabolic form if  $c_1=0$ , reflecting the condition that the slope at the hillcrest is zero.

### **Partial differential equations, PDEs.**

$$\frac{\partial T}{\partial t} = A \frac{\partial^2 T}{\partial x^2} \quad \text{second order, linear PDE}$$

This is the classic diffusion equation, encountered in the study of conducting systems. It has strong analogs in the flow of viscous fluids, and the evolution of topography in the face of diffusive hillslope processes.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = A \frac{\partial^2 u}{\partial x^2} \quad \text{second order, non-linear PDE}$$

These are a few terms of the Navier-Stokes equation describing the conservation of momentum in a 1-dimensional flow dominated by viscous forces. The non-linear term in this case is the second term, with a product of the variable and its derivative.

### **Statistics**

For discrete set of values,  $x_i$ , the **mean** of the population is:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

or for a random variable,  $x$ , with probability of occurrence of the variable  $x_i$  taken to be  $f(x_i)$ ,

$$\bar{x} = \sum_{i=1}^n x_i f(x_i)$$

Note that the probabilities  $f(x)$  are constrained to have  $\sum_n f(x_i) = 1$ .

The **variance** is

$$\text{Var} = \sum_{i=1}^n x_i^2 f(x_i) - \bar{x}^2$$

and the **standard deviation** is the square root of the variance:

$$\sigma = \sqrt{\text{var}}$$

For *continuously* distributed variables, however, we may formalize the statistics using a continuously distributed probability of occurrence. A *probability density function*, or *pdf*, is a function that identifies the probability of occurrence of some event,  $x$ . The pdf,  $f(x)$  is defined as

$$P(a,b) = \int_a^b f(x)dx$$

where  $P(a,b)$  signifies the probability of finding the value between  $a$  and  $b$ .

All *pdfs* are constrained by  $\int_{-\infty}^{\infty} f(x)dx = 1$ ; in other words, the probability that the variable will fall between  $-$  and  $+$ infinity is unity. In addition, one might want to know the cumulative probability density function,  $F(x)$ , defined as

$$F(a) = P(x \leq a) = \int_{-\infty}^a f(x)dx$$

We may now be precise in defining several statistical measures, as shown in **Figure 17**.

The **mode** of a distribution is that value of  $x$  corresponding to the peak in the probability density function, i.e. the most probable value.

The **mean** of a distribution,  $\bar{x}$ , is defined formally to be

$$\bar{x} = \int_{-\infty}^{\infty} x f(x)dx$$

The **median** is that value of  $x$  at which the cumulative probability distribution function is 0.5, i.e.:

$$F(x) = \int_{-\infty}^x f(x)dx = 0.5$$

The spread in the distribution, or its width, is often desired as well. Formally, the **standard deviation** of a distribution is

$$\sigma = \left[ \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)dx \right]^{1/2}$$



and the **variance** is the square of this. Note the standard deviation has the same units as the variable, while the variance has units of the variable squared. This is also called the second moment of the distribution, named for the second power in the formula. (In this terminology, the first moment is the mean, as  $x$  is taken to the first power.)

We encounter a wide range of pdfs in geomorphology, most of which are captured in the following few examples (some shown in [Figure 18](#)). Note that in each case there is an easily identified function with which you are likely already familiar, multiplied by some collection of constants out in front. The role of these constants is to assure that the integral over the full range is 1.

The *uniform distribution*. Here there is an equal, or uniform, probability of finding the function over a specified interval.

$$f(x) = \frac{1}{b-a}; a \leq x \leq b$$

$$f(x) = 0; x < a, x > b$$

The *exponential* distribution. The probability is maximum at  $x=0$  and falls off exponentially to zero at infinity.

$$f(x) = \frac{1}{x^*} e^{-x/x^*}; x \geq 0$$

$$f(x) = 0; x < 0$$

Note that the distribution is defined by a single parameter,  $x^*$ , which sets how rapidly the distribution falls off.

The *gaussian* or *normal* distribution (already illustrated in [Figure 11](#)). This is often encountered in error analysis, as errors are supposed to be normally, or Gaussianly distributed.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_o)^2}{\sigma^2}}$$

The two parameters in the distribution as defined here are the mean,  $x_o$ , and the standard deviation,  $\sigma$ . This defines the classic bell-shaped curve.

The cumulative gaussian pdf, centered on 0, is so commonly encountered that it has been given its own name: the *error function*

$$erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

Similarly, for problems of opposite symmetry, a *complementary error function* is defined as

$$erfc(\eta) = 1 - erf(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

See our discussion of thermal diffusion problems.

The *gamma* distribution. Note that the probability goes to zero at both  $x=0$ , and  $x=\text{infinity}$ .

$$f(x) = \frac{1}{c} x e^{-x/a}$$

A more general gamma distribution allows  $x$  to be taken to some power.

The *log-normal* distribution. This is often encountered in describing grain size distributions (see Ole Barnsdorff-Neilsen's work).

The *Weibull* distribution is defined by three parameters. This is commonly used in describing meteorological data.

$$f(x) = \frac{b}{a} \left( \frac{x-c}{a} \right)^{b-1} e^{-\left(\frac{x-c}{a}\right)^b}; x \geq c$$

$$f(x) = 0; x < c$$

This is similar to the gamma function, but is more flexible (i.e., has more parameters) to allow non-zero centered distributions., and greater than exponential declines in probability at high  $x$ .

The *Binomial* distribution.

The *Poisson* distribution.

$$f(k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Both the mean and the variance of this distribution equal  $\lambda$ , meaning that the standard deviation is  $\sqrt{\lambda}$ .

**Goodness of fit.** We often need to evaluate how well our model fits a particular set of data. One common means employs the chi-square statistic:

$$\chi^2 = \sum_{i=1}^n \left( \frac{x_{mi} - x_{oi}}{\sigma_i} \right)^2$$

in which  $x_m$  corresponds to a modeled value, while  $x_o$  is the corresponding data value. The standard deviation,  $\sigma$ , reflects the expected range or error in our knowledge of the real data, and acts to weight the data. Modeling strategies reduce to a search for the model that best minimizes this statistic.

## **References**

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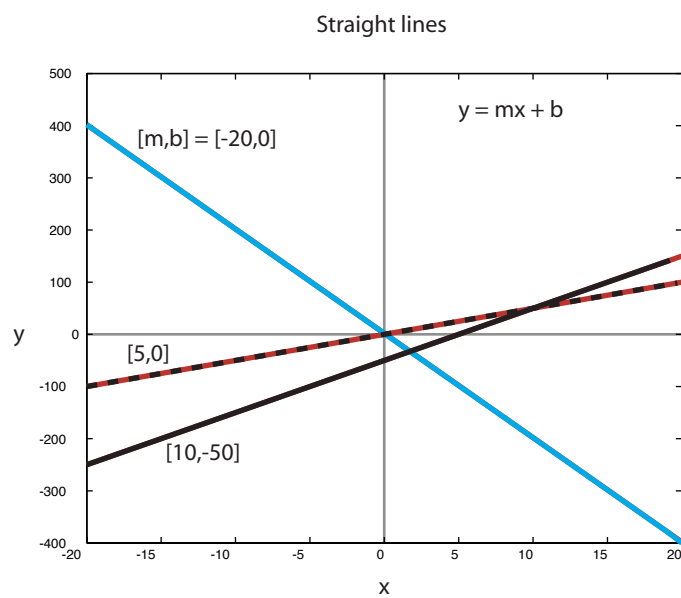


Figure 1. The straight line.

## Negative exponentials

$$y = A e^{-(x/x^*)}$$

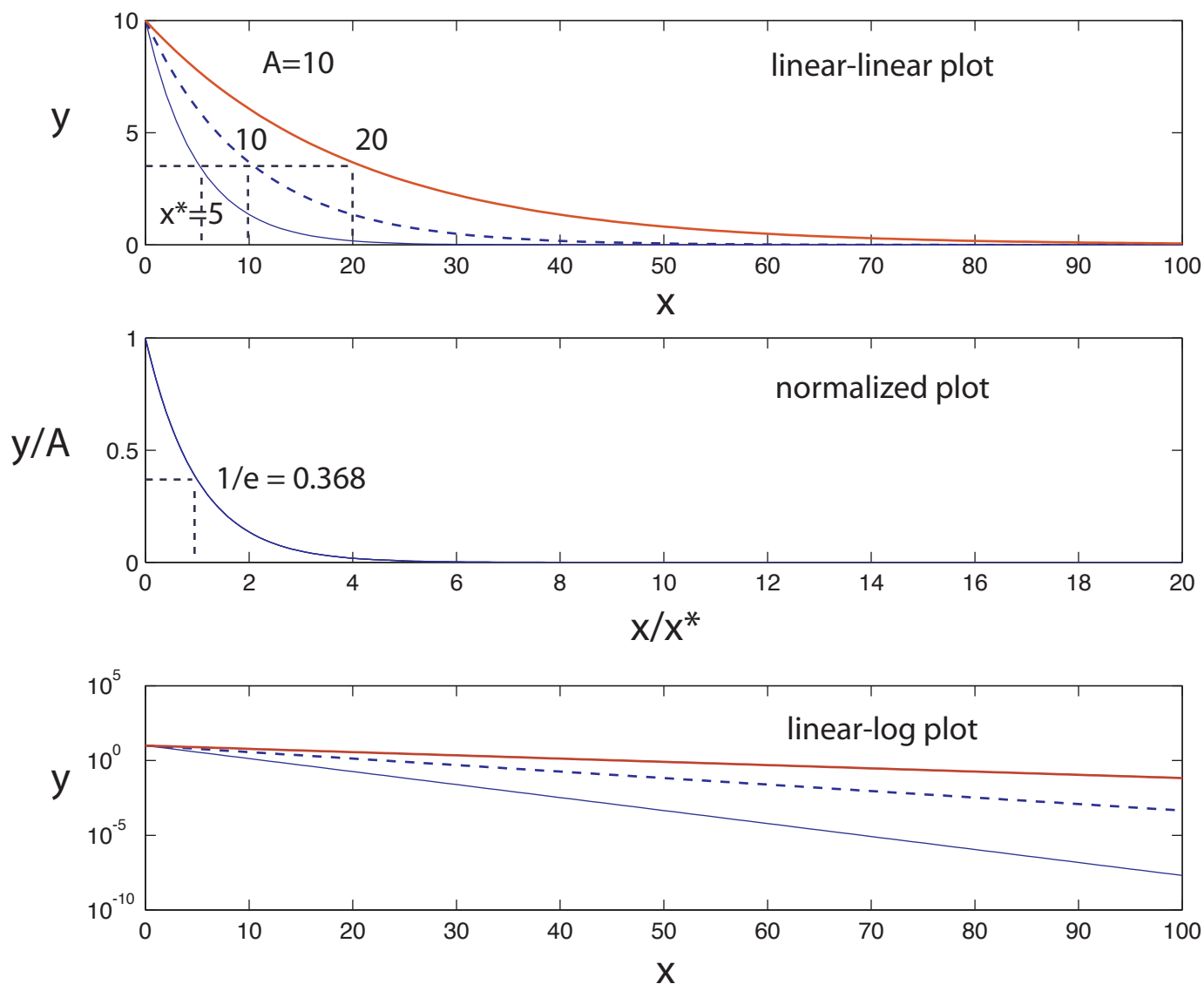


Figure 2. The negative exponential function. This is characterized by two constants, here denoted  $A$ , and  $x^*$ .

### Positive exponential functions

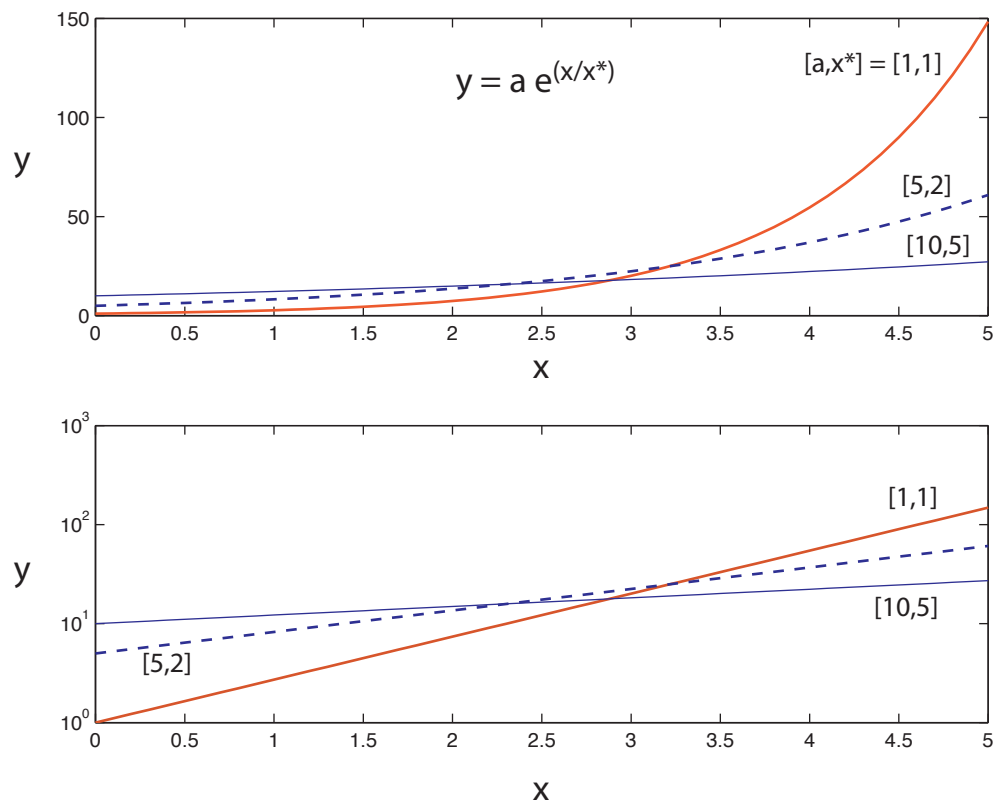


Figure 3. The positive exponential function. This too is characterized by two constants, here denoted  $a$  and  $x^*$ .

$$y = a (1 - e^{-x/x^*})$$

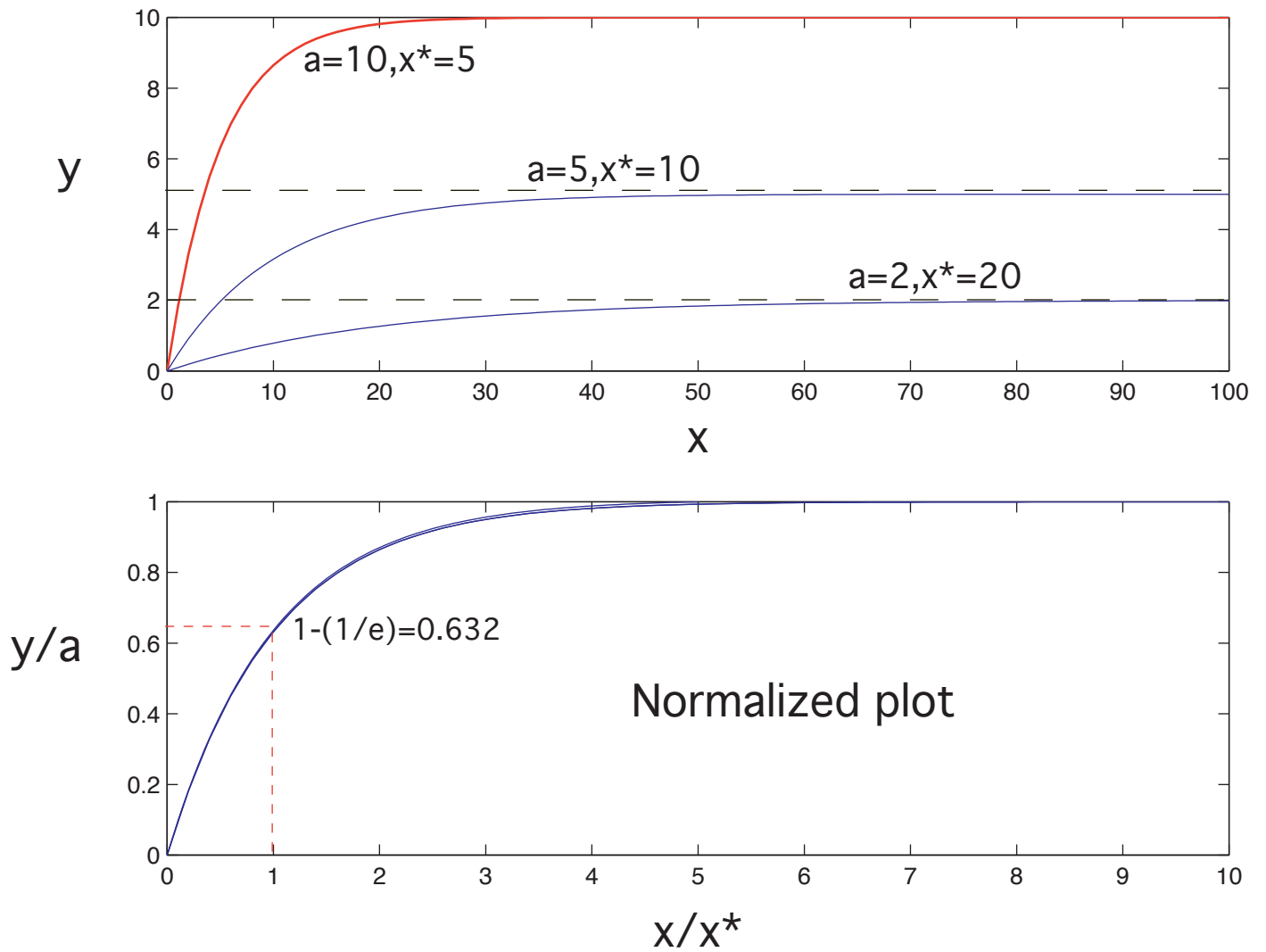


Figure 4. An example of an asymptotic function.

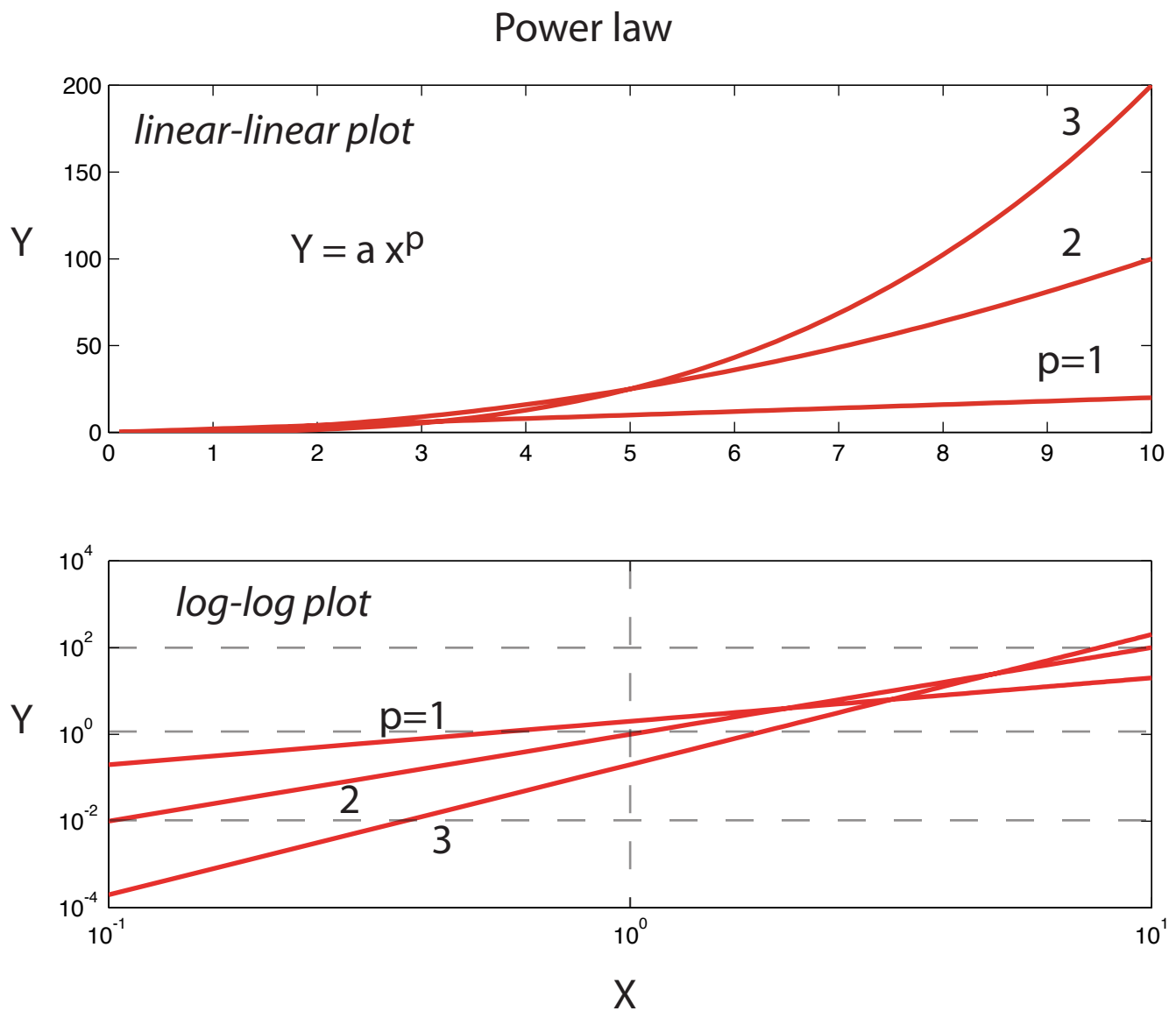


Figure 5. Power law functions, here shown with positive powers. Note that power laws on log-log plots are straight lines.



## Parabolas

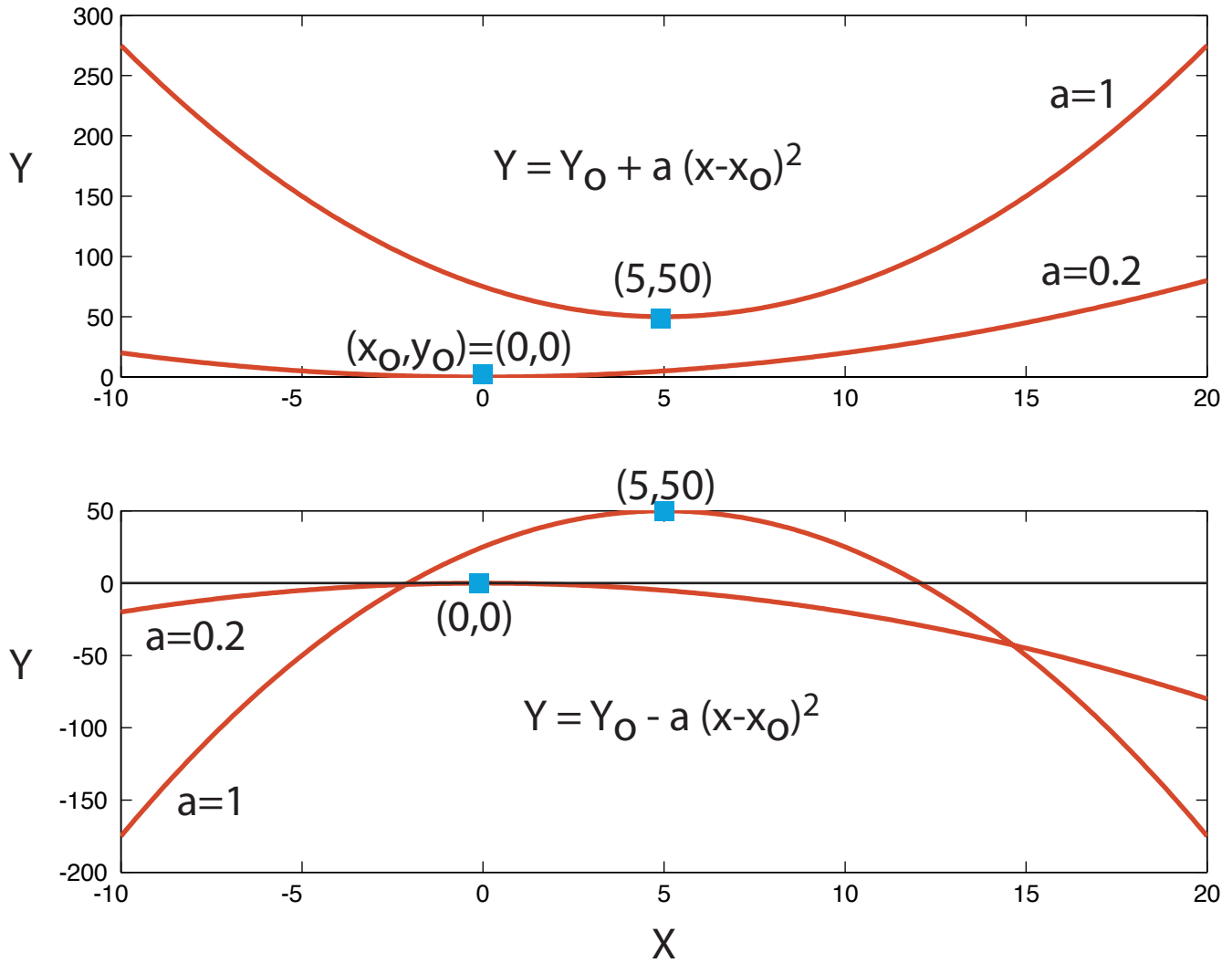


Figure 6. Parabolas. Here examples are shown that are either downward concave or upward convex, and shifted from  $(0,0)$ .

## log functions

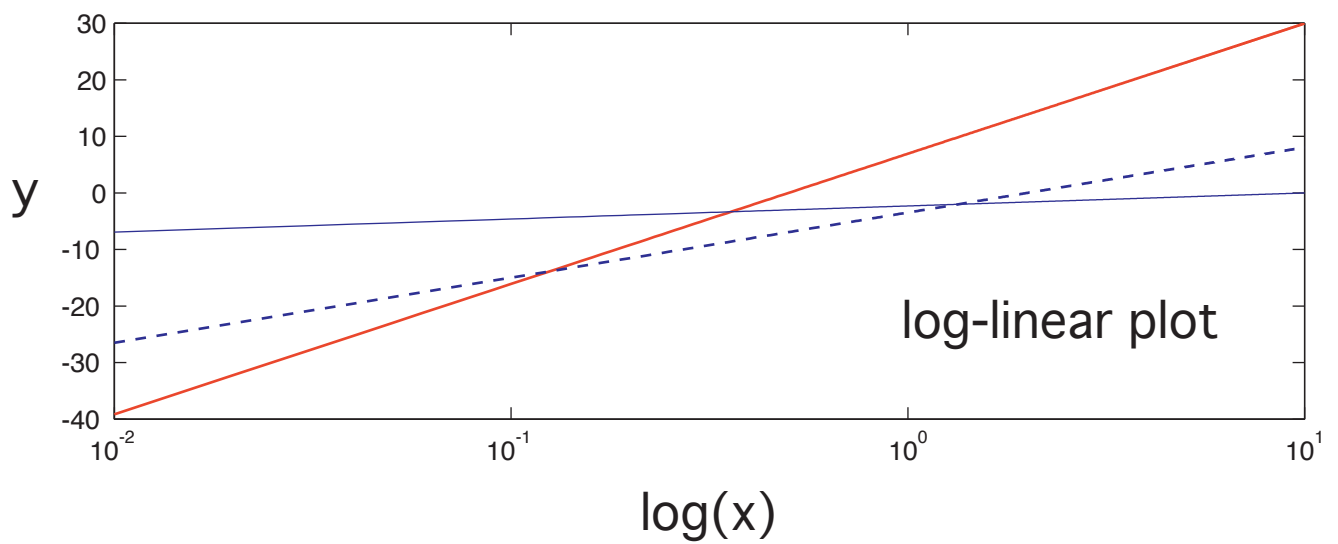
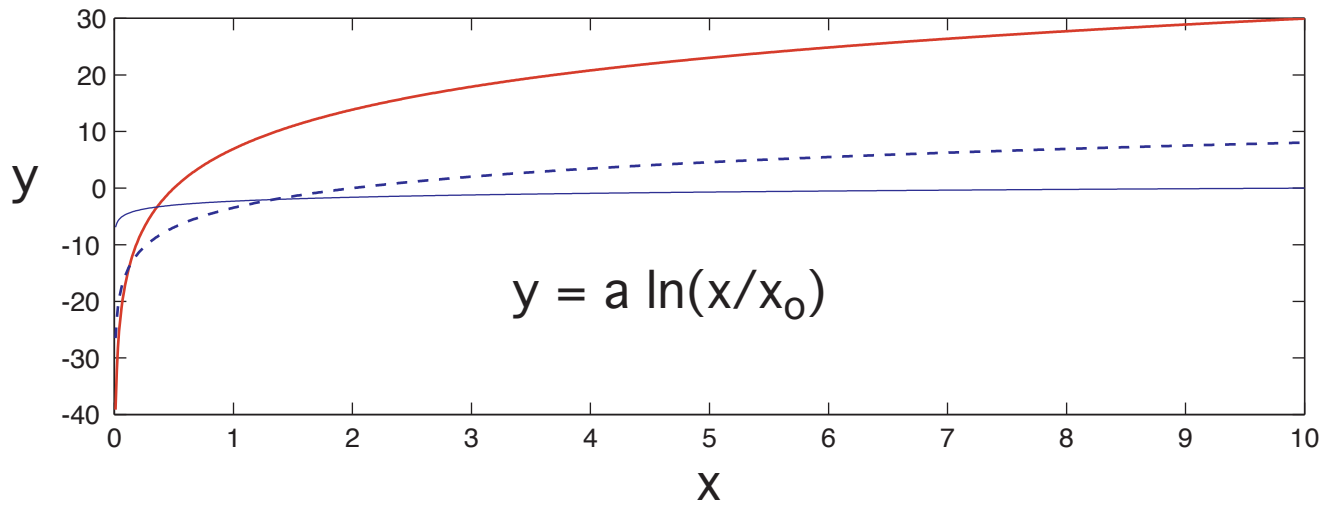


Figure 7. Logarithmic functions. These plot as straight lines on log-linear graphs.

## Sine and cosine

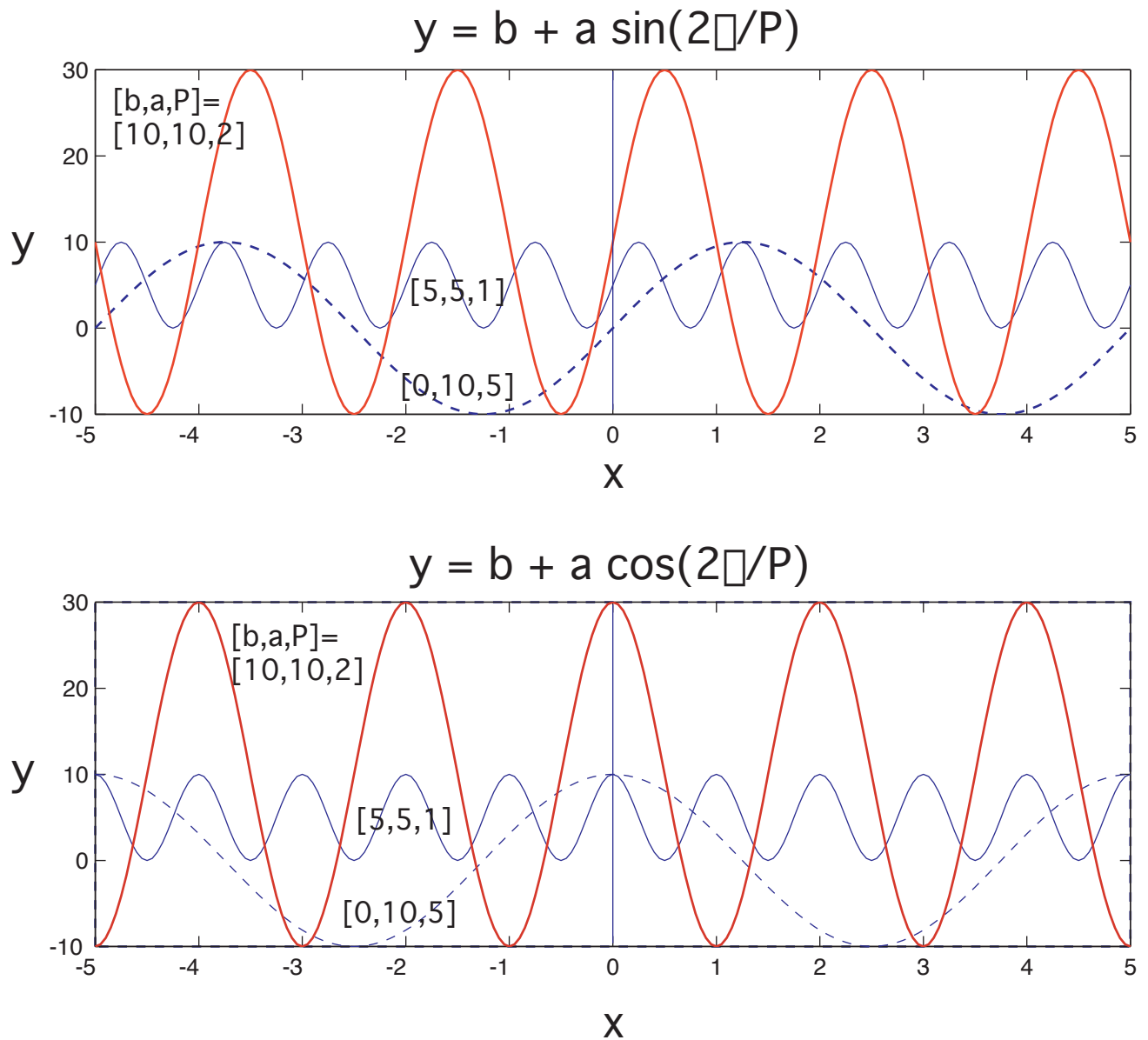


Figure 8. Trigonometric functions sine and cosine. Effects of changing shift,  $b$ , amplitude,  $a$ , and period,  $P$ , are shown.

## Hyperbolic tangents

$$y = b + a \tanh((x-x_0)/x^*)$$

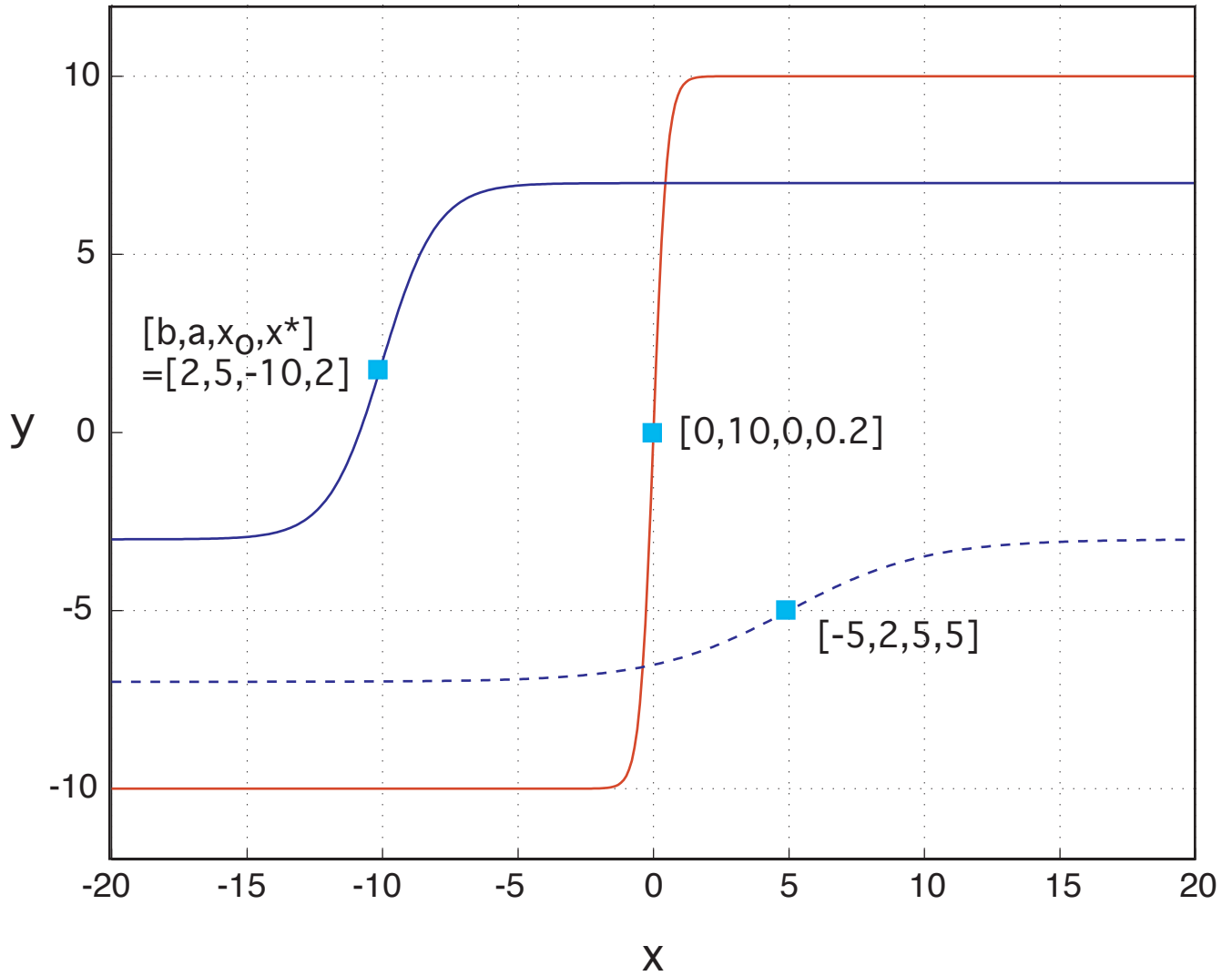


Figure 9. Hyperbolic tangent. Effects of changing shift,  $b$ , amplitude,  $a$ , center,  $x_0$ , and horizontal scale over which the step occurs,  $x^*$ , are shown.

## Hyperbolic sine and cosine

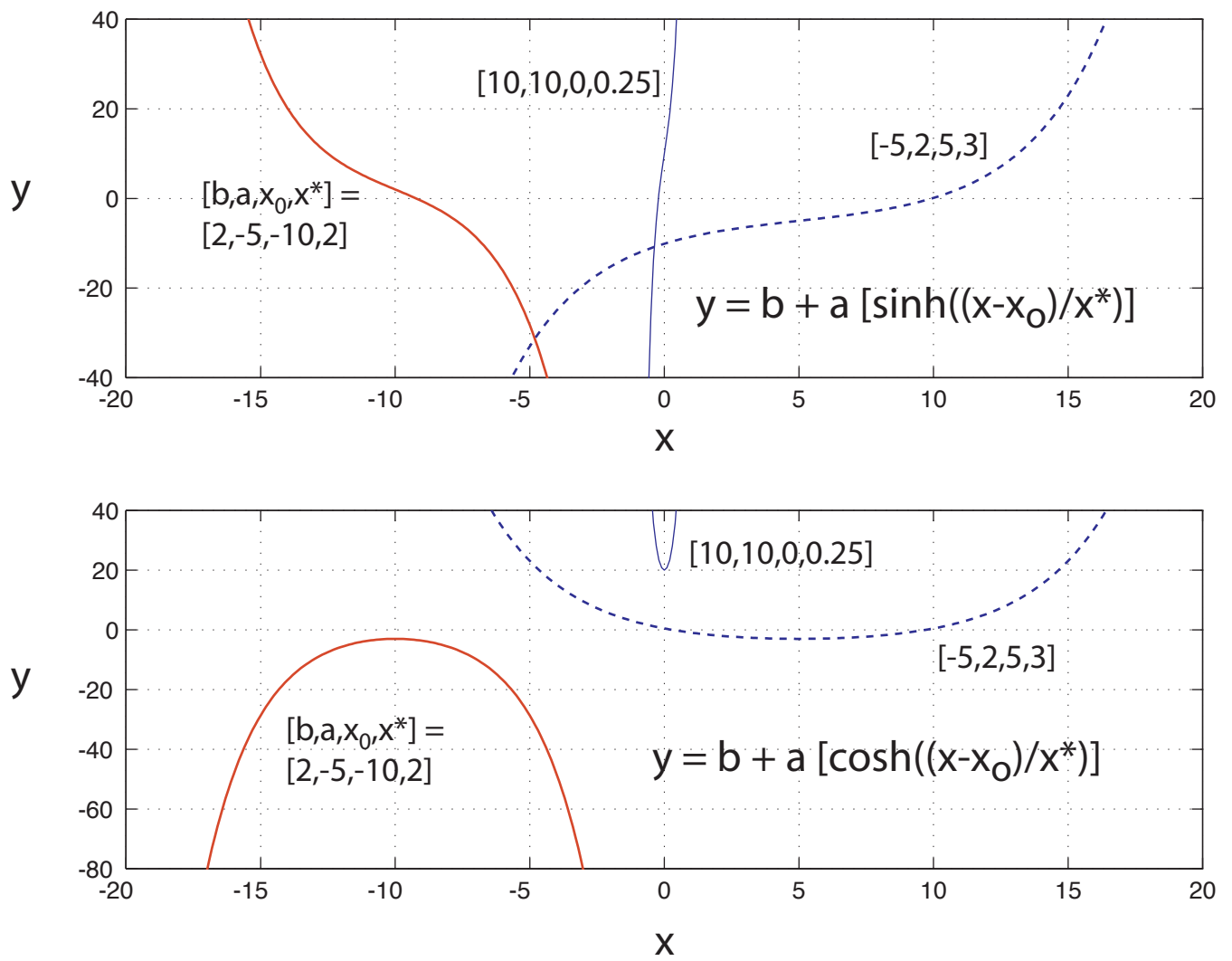


Figure 10. Hyperbolic sine, sinh, and cosine, cosh.

# Gaussian

$$y = a e^{-[(x-x_0)/x^*]^2}$$

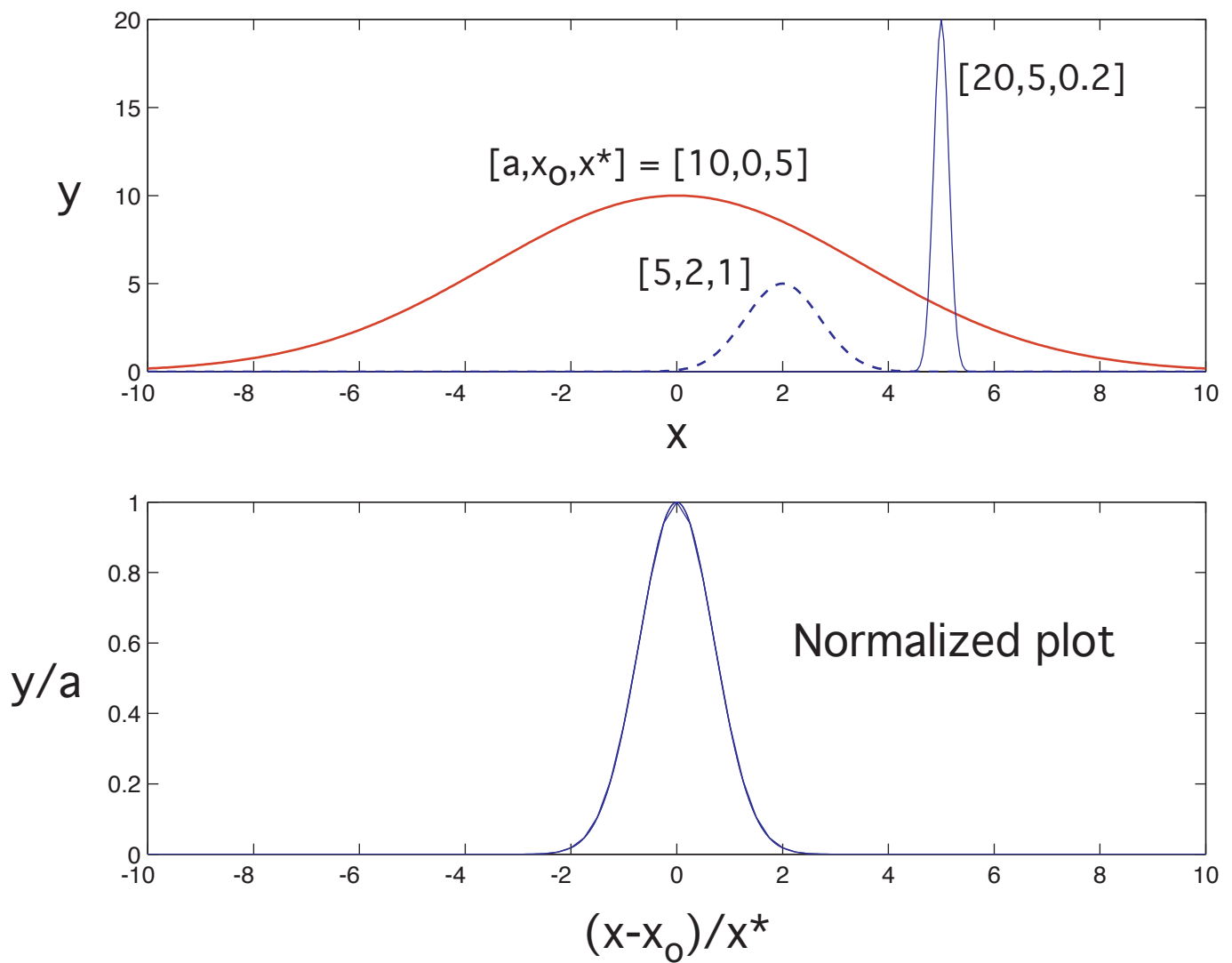
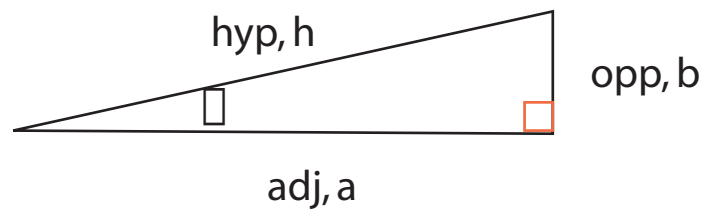


Figure 11. Gaussian function. All plots in the top panel collapse to that on the bottom panel when  $y$  is normalized using  $a$ , and when the  $x$  axis is shifted by  $x_0$  and then scaled by  $x^*$ .



Right triangle. definition sketch for trigonometric functions  
 $\sin = \text{opp}/\text{hyp}$ ;  $\cos = \text{adj}/\text{hyp}$ ;  $\tan = \text{opp}/\text{adj}$

Figure 12. Right triangle. Definitions of sine, cosine and tangent use sides labeled opposite, adjacent and hypotenuse.

## Definition sketch for the derivative

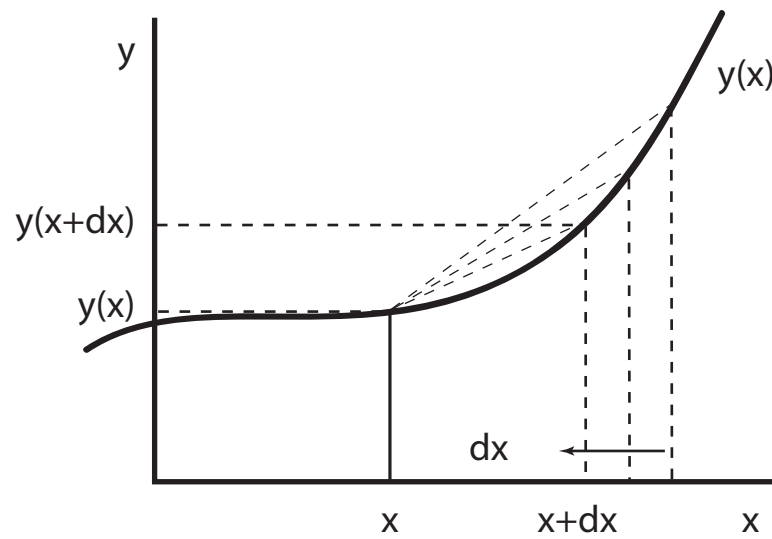


Figure 13. Definition of the derivative,  $dy/dx$ .



## Slope and curvature

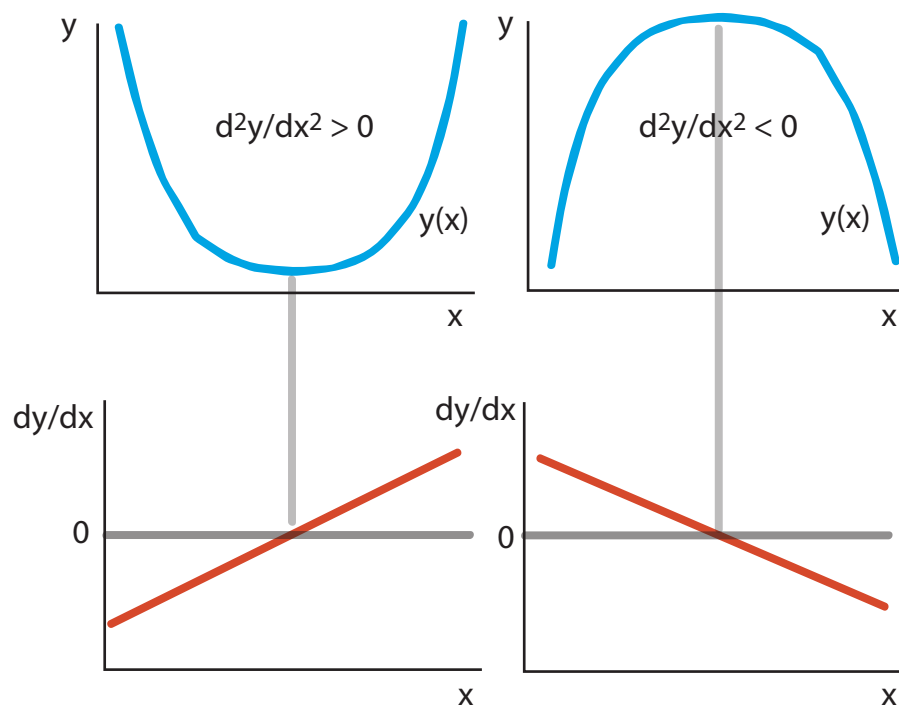


Figure 14. Slope and curvature of the function  $y(x)$ .

Definition sketch for the integral

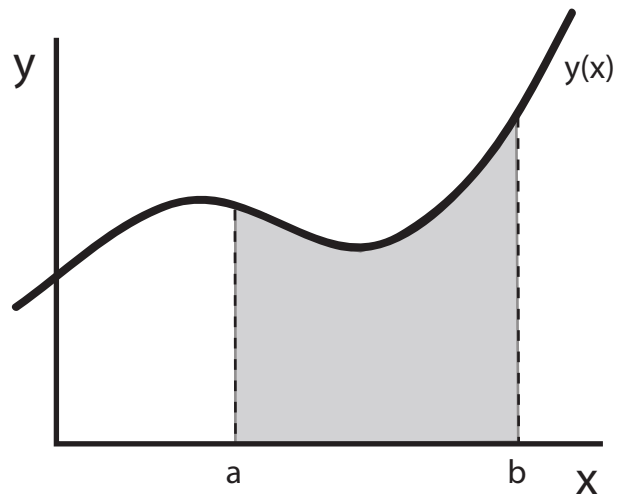


Figure 15. Definition of the definite integral of  $y(x)$  as the area under the curve between  $x=a$  and  $x=b$ .

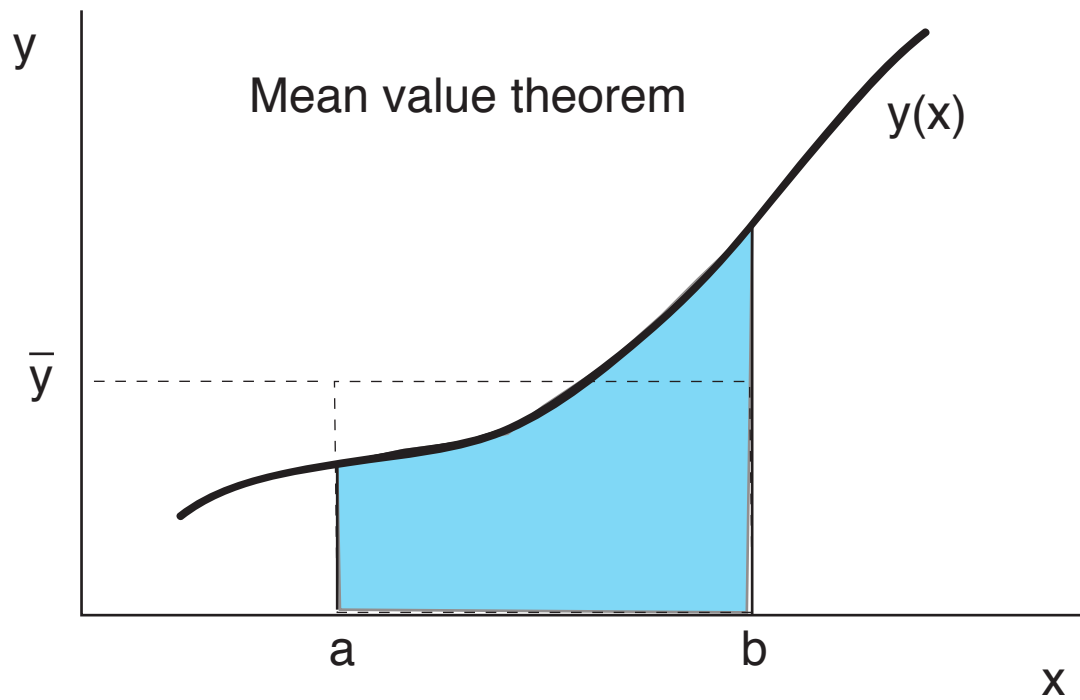


Figure 16. Graphical representation of the mean value theorem used to obtain the mean of the function  $y(x)$  over the interval  $[a, b]$ .

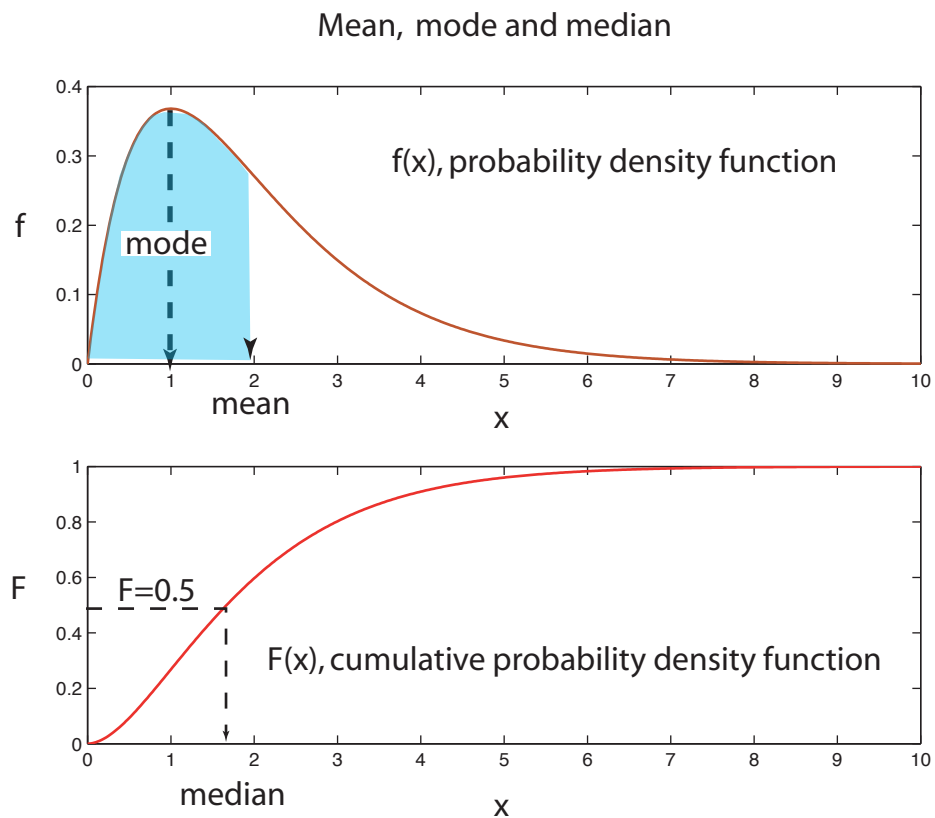


Figure 17. Definitions of mean, mode and median.

## Probability density functions

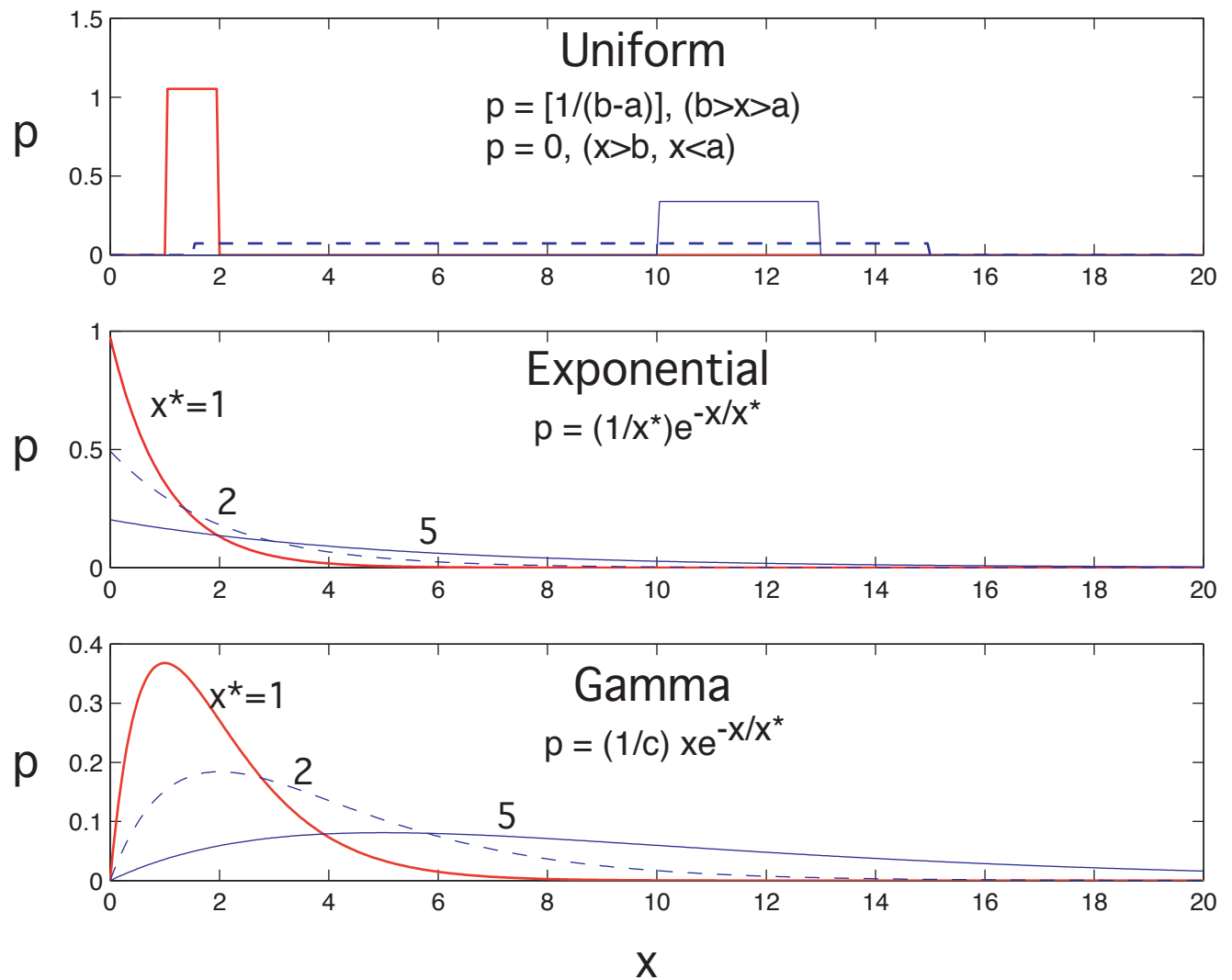


Figure 18. Commonly used probability density functions, pdfs.