The WRF Advection Scheme

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Quick Outline

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1. The Runge-Katta Time Integration Scheme

- Low frequency modes are integrated using a third-order Runge-Katta (RK3) scheme, seen below:

$$\Phi^* = \Phi^t + \frac{\Delta t}{3} R(\Phi^t)$$
$$\Phi^{**} = \Phi^t + \frac{\Delta t}{2} R(\Phi^*)$$
$$\Phi^{t+\Delta t} = \Phi^t + \Delta t R(\Phi^{**})$$

- Φ^*, Φ^{**} are intermediates to get to $\Phi^{t+\Delta t}$
- $R(\Phi^t)$ are the non time-derivative remainders of the governing equations from the model core.

1. The Runge-Katta Time Integration Scheme

Begin Time Step

Begin RK3 Loop: Steps 1, 2, and 3

(1) If RK3 step 1, compute and store F_Φ
(i.e., physics tendencies for RK3 step, including mixing).

(2) Compute $R_\Phi^{t^*}$, (3.13)–(3.18)

Begin Acoustic Step Loop: Steps $1 \rightarrow n$,
RK3 step 1, $n = 1$, $\Delta\tau = \Delta t/3$;
RK3 step 2, $n = n_s/2$, $\Delta\tau = \Delta t/n_s$;
RK3 step 3, $n = n_s$, $\Delta\tau = \Delta t/n_s$.

(3) Advance horizontal momentum, (3.7) and (3.8)
(4) Advance μ_d (3.9) and compute $\Omega''^{\tau+\Delta\tau}$ then advance Θ (3.10)
(5) Advance W and ϕ (3.11) and (3.12)
(6) Diagnose p'' and α'' using (3.5) and (3.4)

End Acoustic Step Loop

(7) Scalar transport: Advance scalars (2.41)
over RK3 substep (3.1), (3.2) or (3.3)
(using mass fluxes U, V and Ω time-averaged over the acoustic steps).

(8) Compute p' and α' using updated prognostic variables in (2.31) and (2.42)

End RK3 Loop

(9) Compute non-RK3 physics (currently microphysics), advance variables.

End Time Step

2. Advection Terms

- Here are the advection equations:

$$
\begin{aligned}
R^{t^*}_{U_{adv}} &= -\, m[\partial_x(Uu) + \partial_y(Vu)] + \partial_\eta(\Omega u) \\
R^{t^*}_{V_{adv}} &= -\, m[\partial_x(Uv) + \partial_y(Vv)] + \partial_\eta(\Omega v) \\
R^{t^*}_{\mu_{adv}} &= -\, m^2[U_x + V_y] + m\Omega_\eta \\
R^{t^*}_{\Theta_{adv}} &= -\, m^2[\partial_x(U\theta) + \partial_y(V\theta)] - m\partial_\eta(\Omega\theta) \\
R^{t^*}_{W_{adv}} &= -\, m[\partial_x(Uw) + \partial_y(Vw)] + \partial_\eta(\Omega w) \\
R^{t^*}_{\phi_{adv}} &= -\, \mu_d^{-1}[m^2(U\phi_x + V\phi_y) + m\Omega\phi_\eta].
\end{aligned}
$$

Again they are solved using their discrete forms.

3. Spatial Differencing Schemes

- Take the discrete equation for scal

$$R^{t^*}_{q_{adv}} = -m^2[\delta_x(U\overline{q}^{x_{adv}}) + \delta_y(V\overline{q}^{y_{adv}})] - m\delta_\eta(\Omega\overline{q}^{\eta_{adv}}).$$

$$\delta_x(U\overline{q}^{x_{adv}}) = \Delta x^{-1}\left[(U\overline{q}^{x_{adv}})_{i+1/2} - (U\overline{q}^{x_{adv}})_{i-1/2}\right].$$

  where:
- The $(q^{x,adv})_{i\,-\,1/2}$ term can be expanded by a Taylor expansion to include second through sixth order terms:
  - Even order operators are spatially centered, and offer no implicit diffusion
  - Odd order operators are not spatially centered, and will inherently diffuse.

4. Stability of the RK3 Scheme

TABLE 1. Maximum stable Courant number for one-dimensional linear advection. Here, U indicates the scheme is unstable.

	Spatial order			
Time scheme	3rd	4th	5th	6th
Leapfrog	U	0.72	U	0.62
RK2	0.88	U	0.30	U
RK3	1.61	1.26	1.42	1.08

- RK3 scheme allows for larger Courant number (i.e., larger stable grid spacing), than corresponding RK2 or leapfrog schemes.

5. a) Advection tests – One dimensional

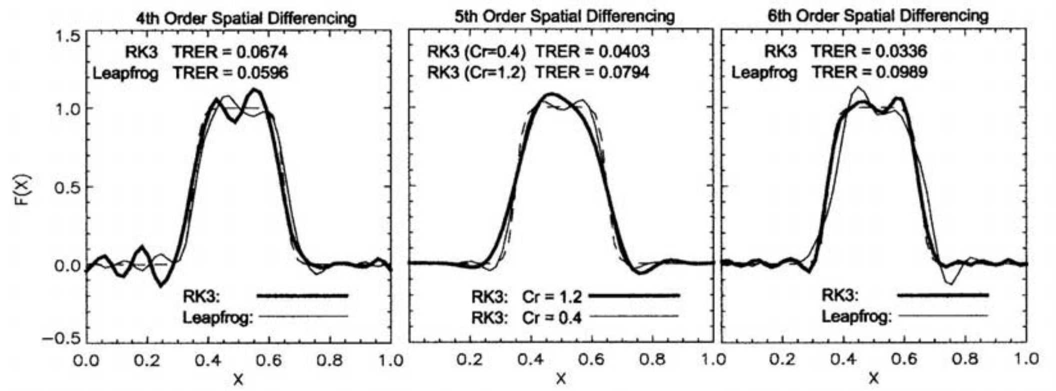

FIG. 1. One-dimensional advection tests for RK3 and leapfrog integration schemes using (a) 4th, (b) 5th, and (c) 6th order spatial discretization schemes. TRER errors for each solution are listed at the top of each box. Unless otherwise noted, the Courant number equals 0.4

- Tested using a smooth square pulse
- RK3 improves with higher order Taylor
- Leapfrog worse at 6th order, than 4th
- Not surprising, lower Courant numbers more accurate.
- There are implicit filters present in RK3

5. b) Advection tests – Two dimensional

TABLE 2. TRER errors and convergence rates for the rotating Gaussian cone problem using the RK3 and leapfrog integration schemes. Results are shown for fourth-, fifth-, and sixth-order spatial differencing. TRER errors are shown at the top of the cell, while convergence rates are listed in italics at the bottom of each cell.

	Scheme				
Resolution	LF-4	RK3-4	RK3-5	RK3-6	LF-6
050 × 050	0.541×10^{-1}	0.598×10^{-1}	0.247×10^{-1}	0.152×10^{-1}	0.116×10^{-1}
100 × 100	0.366×10^{-2} *(3.89)*	0.530×10^{-2} *(3.49)*	0.158×10^{-2} *(3.96)*	0.412×10^{-3} *(5.20)*	0.207×10^{-2} *(2.48)*
200 × 200	0.339×10^{-3} *(3.43)*	0.344×10^{-3} *(3.95)*	0.749×10^{-4} *(4.40)*	0.324×10^{-4} *(3.67)*	0.565×10^{-3} *(1.88)*
400 × 400	0.125×10^{-3} *(1.45)*	0.219×10^{-4} *(3.97)*	0.527×10^{-5} *(3.83)*	0.402×10^{-5} *(3.01)*	0.142×10^{-3} *(1.99)*
800 × 800	0.344×10^{-4} *(1.86)*	0.144×10^{-4} *(3.93)*	0.503×10^{-6} *(3.39)*	0.503×10^{-6} *(3.00)*	0.355×10^{-4} *(2.00)*

- Tested using a Gaussian cone in a square domain.
- Low resolutions, RK3-6 and LF6 have smallest errors; high resolutions, LF has 70x larger errors than RK3-5 and RK3-6 (faster convergence rate)

6. Conclusions from Wicker and Skamarock

- Maximum stable Courant numbers near 1 for three dimensional flows (1.61 for 1-D), but need the three iterations to complete a time step.
- They believe RK3 represents the best combo of accuracy and simplicity, with the largest possible time step.
  - 3-D testing shows time steps up to 2x larger than corresponding leapfrog based time-split models.