PRESSURE GRADIENT FORCE OF WRF

Jih-Wang Wang

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Colorado State University
Atmospheric Science Department
Fort Collins, CO

Hydrostatically-balanced Perturbation Form of the Governing Equations

$$\frac{\partial U}{\partial t} + m\left[\frac{\partial}{\partial x}(Uu) + \frac{\partial}{\partial y}(Vu)\right] + \frac{\partial}{\partial \eta}(\Omega u) + (\mu_d \alpha \frac{\partial p'}{\partial x} + \mu_d \alpha' \frac{\partial \overline{p}}{\partial x})$$
$$+ (\alpha / \alpha_d)(\mu_d \frac{\partial \phi'}{\partial x} + \frac{\partial p'}{\partial \eta}\frac{\partial \phi}{\partial x} - \mu'_d \frac{\partial \phi}{\partial x}) = F_U \quad (2.35)$$

$$\frac{\partial \mu'_d}{\partial t} + m^2(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}) + m\frac{\partial \Omega}{\partial \eta} = 0 \quad (2.38)$$

→ non-hydrostatic terms

$$\frac{\partial \phi'}{\partial t} + \frac{1}{\mu_d}\left[m^2(U\frac{\partial \phi}{\partial x} + V\frac{\partial \phi}{\partial y}) + m\Omega\frac{\partial \phi}{\partial \eta} - gW\right] = 0 \quad (2.39)$$

$$\frac{\partial \Theta}{\partial t} + m^2\left[\frac{\partial}{\partial x}(U\theta) + \frac{\partial}{\partial y}(V\theta)\right] + m\frac{\partial}{\partial \eta}(\Omega\theta) = F_\Theta \quad (2.40)$$

$$p = p_0(R_d\theta_m / p_0\alpha_d)^\gamma \quad (2.31)$$

Assumption

- Hydrostatic (i.e., reference state)
- No map projection factor for simplicity
- Only x direction
- No external forcing (i.e., no physics or parameterization)
- Dry atmosphere
- Reference state horizontally homogeneous

Simplified Governing Equations

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(Uu) + \overline{\mu}\alpha \frac{\partial p'}{\partial x} + \overline{\mu} \frac{\partial \phi'}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial \phi'}{\partial t} + \frac{U}{\overline{\mu}} \frac{\partial \phi'}{\partial x} = 0 \qquad (2)$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial}{\partial x}(U\theta) = 0 \qquad (3)$$

$$p = p_0 (R_d \theta / p_0 \alpha)^{\gamma} \qquad (4)$$

A. Solve for geopotential height

To linearize Equation (2), assume $U = [U] + U''$ *and* $[U] >> U''$

$$\frac{\partial \phi'}{\partial t} + \frac{[U]}{\overline{\mu_d}} \frac{\partial \phi'}{\partial x} = 0$$

Rewrite the equation in discrete form of 2nd order in space and Runge-Kutta 3rd in time

$$\phi_n'^{*} = \phi_n'^{\tau} + \frac{[U]\Delta t}{6\overline{\mu}\Delta x}(\phi_{n-1}'^{\tau} - \phi_{n+1}'^{\tau})$$

$$\phi_n'^{**} = \phi_n'^{\tau} + \frac{[U]\Delta t}{4\overline{\mu}\Delta x}(\phi_{n-1}'^{*} - \phi_{n+1}'^{*})$$

$$\phi_n'^{\tau+1} = \phi_n'^{\tau} + \frac{[U]\Delta t}{2\overline{\mu}\Delta x}(\phi_{n-1}'^{**} - \phi_{n+1}'^{**})$$

$$\phi_n'^{\tau+1} = \phi_n'^{\tau} + \frac{[U]\Delta t}{2\overline{\mu}\Delta x}(\phi_{n-1}'^{\tau} - \phi_{n+1}'^{\tau})$$

$$+ \frac{[U]^2(\Delta t)^2}{8\overline{\mu}^2(\Delta x)^2}(\phi_{n-2}'^{\tau} - 2\phi_n'^{\tau} + \phi_{n+2}'^{\tau})$$

$$+ \frac{[U]^3(\Delta t)^3}{48\overline{\mu}^3(\Delta x)^3}(\phi_{n-3}'^{\tau} - 3\phi_{n-1}'^{\tau} + 3\phi_{n+1}'^{\tau} - \phi_{n+3}'^{\tau})$$

Let $\phi_n'^{\tau} = \hat{\phi}\exp\left[i(kn\Delta x + \omega\tau\Delta t)\right]$

$$\psi^1 = 1 + \frac{[U]\Delta t}{2\overline{\mu}\Delta x}(\psi_{-1} - \psi_1) + \frac{[U]^2(\Delta t)^2}{8\overline{\mu}^2(\Delta x)^2}(\psi_{-2} - 2 + \psi_2)$$

$$+ \frac{[U]^3(\Delta t)^3}{48\overline{\mu}^3(\Delta x)^3}(\psi_{-3} - 3\psi_{-1} + 3\psi_1 - \psi_3)$$

$$\lambda\cos(\omega_r\Delta t) = 1 - \frac{\overline{U}^2(\Delta t)^2}{4\overline{\mu}^2(\Delta x)^2} + \frac{\overline{U}^2(\Delta t)^2}{4\overline{\mu}^2(\Delta x)^2}\cos(2k\Delta x)$$

$$= 1 - \frac{\gamma^2}{4} + \frac{\gamma^2}{4}\cos(2k\Delta x)$$

$$\lambda\sin(\omega_r\Delta t) = -\frac{\overline{U}\Delta t}{\overline{\mu}\Delta x}\sin(k\Delta x) - \frac{\overline{U}^3(\Delta t)^3}{24\overline{\mu}^3(\Delta x)^3}\sin(3k\Delta x) + \frac{\overline{U}^3(\Delta t)^3}{8\overline{\mu}^3(\Delta x)^3}\sin(k\Delta x)$$

$$= -\gamma\sin(k\Delta x) - \frac{\gamma^3}{24}\sin(3k\Delta x) + \frac{\gamma^3}{8}\sin(k\Delta x)$$

$$\lambda^2 = 1 + \frac{\gamma^4}{16} + \frac{\gamma^4}{16}\cos^2(2k\Delta x) - \frac{\gamma^2}{2} + \frac{\gamma^4}{8}\cos(2k\Delta x) + \frac{\gamma^2}{2}\cos(2k\Delta x)$$

$$+ \gamma^2\sin^2(k\Delta x) + \frac{\gamma^6}{576}\sin^2(3k\Delta x) + \frac{\gamma^6}{64}\sin^2(k\Delta x) + \frac{\gamma^4}{12}\sin(k\Delta x)\sin(3k\Delta x)$$

$$- \frac{\gamma^6}{96}\sin(k\Delta x)\sin(3k\Delta x) - \frac{\gamma^4}{4}\sin^2(k\Delta x)$$

$$= 1 + \frac{\gamma^4}{16} + \frac{\gamma^4}{16}\cos^2(2k\Delta x) - \frac{\gamma^4}{8}\cos(2k\Delta x)$$

$$+ \frac{\gamma^6}{576}\sin^2(3k\Delta x) + \frac{\gamma^6}{64}\sin^2(k\Delta x) + \frac{\gamma^4}{12}\sin(k\Delta x)\sin(3k\Delta x)$$

$$- \frac{\gamma^6}{96}\sin(k\Delta x)\sin(3k\Delta x) - \frac{\gamma^4}{4}\sin^2(k\Delta x)$$

A. If wavelength is $2\Delta x$

$$\lambda^2 = 1$$

$$c / c_a = 0$$

B. If wavelength is $4\Delta x$

$$\lambda^2 = 1 - \frac{\gamma^4}{12} + \frac{\gamma^6}{36}$$

$$c / c_a = \frac{\bar{\mu}}{\Delta t k[U]} \cos^{-1}\left[\frac{1}{\lambda}(1 - \frac{\gamma^2}{2})\right] = \frac{2}{\pi\gamma} \cos^{-1}\left[\frac{1}{\lambda}(1 - \frac{\gamma^2}{2})\right]$$

C. If wavelength is $12\Delta x$

$$\lambda^2 = 1 - \frac{\gamma^4}{192} + \frac{\gamma^6}{2304}$$

$$c / c_a = \frac{\bar{\mu}}{\Delta t k[U]} \cos^{-1}\left[\frac{1}{\lambda}(1 - \frac{\gamma^2}{8})\right] = \frac{6}{\pi\gamma} \cos^{-1}\left[\frac{1}{\lambda}(1 - \frac{\gamma^2}{8})\right]$$

D. If wavelength is $24\Delta x$

$$\lambda^2 = 1 - 3.74 \times 10^{-4} \gamma^4 + 8.35 \times 10^{-6} \gamma^6$$

$$c / c_a = \frac{\bar{\mu}}{\Delta t k[U]} \cos^{-1}\left[\frac{1}{\lambda}(1 - \frac{(2-\sqrt{3})\gamma^2}{8})\right] = \frac{12}{\pi\gamma} \cos^{-1}\left[\frac{1}{\lambda}(1 - \frac{(2-\sqrt{3})\gamma^2}{8})\right]$$

B. Solve for potential temperature

Assume basic state potential temperature horizontally homogeneous

$$\frac{\partial \theta''}{\partial t} + \frac{1}{\overline{\mu}} \frac{\partial}{\partial x}(U\theta'') = 0$$

Linearize with respect to momentum, just like geopotential height

$$\frac{\partial \theta''}{\partial t} + \frac{[U]}{\overline{\mu}} \frac{\partial}{\partial x}\theta'' = 0$$

Solution is the same as for geopotential height

$$\theta'' = \hat{\theta} \exp[i(kn\Delta x + \omega\tau\Delta t)]$$

C. Solve for momentum

Consider only external waves

$$p = p_0 (R_d \theta / p_0 \alpha)^{\gamma}$$

$$\Rightarrow \quad \frac{\partial p'}{\partial x} = p_0 (\frac{R_d}{p_0 \alpha})^{\gamma} \frac{\partial \theta''^{\gamma}}{\partial x}$$

$$\frac{\partial U}{\partial t} + \overline{\mu} \alpha \frac{\partial p'}{\partial x} + \overline{\mu} \frac{\partial \phi'}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial U''}{\partial t} + \overline{\mu} \alpha p_0 (\frac{R_d}{p_0 \alpha})^{\gamma} \frac{\partial \theta''^{\gamma}}{\partial x} + \overline{\mu} \frac{\partial \phi'}{\partial x} = 0$$

$$U''^{\tau+T}_{n+1/2} = A(\lambda^{\gamma})^{T} \exp\{i[k\gamma(n+1/2)\Delta x + \omega_r \gamma(\tau+T)\Delta t]\} + B\lambda^{T} \exp\{i[k\gamma(n+1/2)\Delta x + \omega_r \gamma(\tau+T)\Delta t]\}$$

Phase speed is the same as potential temperature and geopotential height. However, the amplitude change differs …

Summary for Simplified & Linearized Governing Equations

- 2nd order in space and Runge-Kutta 3rd in time preserves amplitude or nearly preserves amplitude in advection.
- Short waves travel more slowly. 2-delta_x waves are stationary, while long waves travel almost as actual speed. Bad for short waves!
- The influence of thermodynamic energy on momentum is to the power of Cp/Cv, and acts on different spectrum.
- Note: it's only 2nd order in space, usually, people use 4th order or higher
- Reminder: a. Here we didn't consider non-linear problem and aliasing; b. Linearization breaks flux forms of advection.

If we don't linearize the equations, the (1)-(4) system is still solvable.

Step 1:

$$U''^{*}_{n+1/2} = U''^{\tau}_{n+1/2} + \frac{\overline{\mu}\Delta t}{3\Delta x}\left[\alpha p_0 \left(\frac{R_d}{p_0 \alpha}\right)^{\gamma} \left(\theta''^{\gamma}\Big|_{n}^{\tau} - \theta''^{\gamma}\Big|_{n+1}^{\tau}\right) + \left(\phi'^{\tau}_{n} - \phi'^{\tau}_{n+1}\right)\right]$$

$$\phi'^{*}_{n} = \phi'^{\tau}_{n} + \frac{\Delta t}{12\overline{\mu}\Delta x}\left[(U''^{\tau}_{n-1/2} + U''^{\tau}_{n+1/2})(\phi'^{\tau}_{n-1} - \phi'^{\tau}_{n+1})\right]$$

$$\theta''^{*}_{n} = \theta''^{\tau}_{n} + \frac{\Delta t}{12\overline{\mu}\Delta x}\left[(U''^{\tau}_{n-3/2} + U''^{\tau}_{n-1/2})\theta''^{\tau}_{n-1} - (U''^{\tau}_{n+1/2} + U''^{\tau}_{n+3/2})\theta''^{\tau}_{n+1}\right]$$

Step 2:

$$U''^{**}_{n+1/2} = U''^{\tau}_{n+1/2} + \frac{\overline{\mu}\Delta t}{2\Delta x}\left[\alpha p_0 \left(\frac{R_d}{p_0 \alpha}\right)^{\gamma} \left(\theta''^{\gamma}\Big|_n^* - \theta''^{\gamma}\Big|_{n+1}^*\right) + (\phi'^*_n - \phi'^*_{n+1})\right]$$

$$\phi'^{**}_n = \phi'^{\tau}_n + \frac{\Delta t}{8\overline{\mu}\Delta x}\left[(U''^*_{n-1/2} + U''^*_{n+1/2})(\phi'^*_{n-1} - \phi'^*_{n+1})\right]$$

$$\theta''^{**}_n = \theta''^{\tau}_n + \frac{\Delta t}{8\overline{\mu}\Delta x}\left[(U''^*_{n-3/2} + U''^*_{n-1/2})\theta''^*_{n-1} - (U''^*_{n+1/2} + U''^*_{n+3/2})\theta''^*_{n+1}\right]$$

Step 3:

$$U''^{\tau+1}_{n+1/2} = U''^{\tau}_{n+1/2} + \frac{\overline{\mu}\Delta t}{\Delta x}\left[\alpha p_0 \left(\frac{R_d}{p_0 \alpha}\right)^{\gamma} \left(\theta''^{\gamma}\Big|_{n}^{**} - \theta''^{\gamma}\Big|_{n+1}^{**}\right) + \left(\phi'^{**}_{n} - \phi'^{**}_{n+1}\right)\right]$$

$$\phi'^{\tau+1}_{n} = \phi'^{\tau}_{n} + \frac{\Delta t}{4\overline{\mu}\Delta x}\left[(U''^{**}_{n-1/2} + U''^{**}_{n+1/2})(\phi'^{**}_{n-1} - \phi'^{**}_{n+1})\right]$$

$$\theta''^{\tau+1}_{n} = \theta''^{\tau}_{n} + \frac{\Delta t}{4\overline{\mu}\Delta x}\left[(U''^{**}_{n-3/2} + U''^{**}_{n-1/2})\theta''^{**}_{n-1} - (U''^{**}_{n+1/2} + U''^{**}_{n+3/2})\theta''^{**}_{n+1}\right]$$

Let

$$U''^{\tau}_{n+1/2} = \hat{U}_1 \exp\{i[k\gamma(n+1/2)\Delta x + \omega\gamma\tau\Delta t]\} + \hat{U}_2 \exp\{i[k(n+1/2)\Delta x + \omega\tau\Delta t]\}$$

$$\theta''^{\tau}_{n} = \hat{\theta} \exp[i(kn\Delta x + \omega\tau\Delta t)]$$

$$\phi'^{\tau}_{n} = \hat{\phi} \exp[i(kn\Delta x + \omega\tau\Delta t)]$$

You can try it …

		$\gamma = \frac{[U]\Delta t}{\overline{\mu}\Delta x}$		
		0.1	0.5	0.9
wavelength	$2\Delta x$	$\lambda = 1$ $c / c_a = 0$	$\lambda = 1$ $c / c_a = 0$	$\lambda = 1$ $c / c_a = 0$
	$4\Delta x$	$\lambda \cong 1$ $c / c_a = 0.637$	$\lambda = 0.995$ $c / c_a = 0.632$	$\lambda = 0.960$ $c / c_a = 0.638$
	$12\Delta x$	$\lambda \cong 1$ $c / c_a = 0.955$	$\lambda \cong 1$ $c / c_a = 0.953$	$\lambda \cong 0.997$ $c / c_a = 0.949$
	$24\Delta x$	$\lambda \cong 1$ $c / c_a = 0.989$	$\lambda \cong 1$ $c / c_a = 0.988$	$\lambda \cong 1$ $c / c_a = 0.987$