ERROR REPORTED BY JERRY DAVIS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC



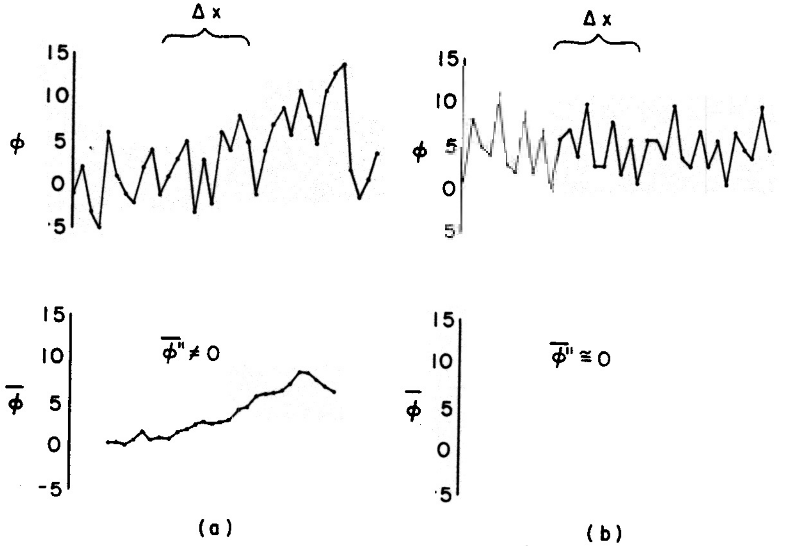


**Fig. 4-2.** A one-dimensional schematic illustration of a situation where (a) $\overline{\phi''} \neq 0$ and (b) $\overline{\phi''} \simeq 0$. The averaging length is illustrated by the interval $\Delta x$ drawn on the figure ($\bar{\phi} = \frac{1}{10}\int_{x_i}^{x_i+10} \phi\, dx$). Since $\phi = \bar{\phi} + \phi''$, $\overline{\phi''} = 0$ only if $\bar{\phi} = \bar{\bar{\phi}}$.

Even with the simplifications, Eq. (4-7) contains two additional terms not found in Eq. (2-45) that involve the correlation between the subgrid-scale variables. The second of these terms, $\overline{\alpha'' \partial p''/\partial x_i}$, could be eliminated if the assumption is made in Eq. (4-4) that $|\alpha''|/\bar{\alpha} \simeq |\alpha''|/\alpha_0 \ll 1$ [In Section 3.1, $\alpha_0$ was defined as a synoptic-scale specific volume in the derivation of the approximate forms of the conservation-of-mass relation. The mathematical definition of this synoptic scale is given by Eq. (4-12).]

With this requirement on specific volume and the assumption that Eq. (3-11) can be written as

$$\frac{\partial}{\partial x_j} \rho_0 u_j \simeq \frac{\partial}{\partial x_j} \bar{\rho} u_j \quad \left(\text{since } \frac{|\alpha''|}{\bar{\alpha}} \simeq \frac{|\alpha''|}{\alpha_0} \ll 1\right), \tag{4-9}$$

using the simplifying assumptions given by Eq. (4-8), Eq. (4-5) can be written as

CORRECTION: Should be p not rho

$$\bar{\rho}\frac{\partial \bar{u}_i}{\partial t} = -\frac{\partial}{\partial x_j}\bar{\rho}\,\bar{u}_j\,\bar{u}_i - \frac{\partial}{\partial x_j}\bar{\rho}\overline{u_j'' u_i''} - \frac{\partial \bar{p}}{\partial x_i} - \bar{\rho} g \delta_{i3} - 2\epsilon_{ijk}\Omega_j \bar{u}_k \bar{\rho}, \tag{4-10}$$

where, since $|\alpha''|/\bar{\alpha} \ll 1$, the pressure gradient term is represented by $\bar{\alpha}\,\partial\bar{p}/\partial x$. The remaining *subgrid-scale correlation term*, $\bar{\rho}\overline{u_j'' u_i''}$, represents the contributions of the smaller scales on the resolvable grid scale resulting from fluctuating velocity components and is in general very important in all aspects of dynamic

Reported by Bob Bornstein, San Jose State University, 3/27/03

are valid only over spatial and temporal intervals that are much smaller than the mesoscale space and time scales.

The functional form of this generalized vertical coordinate transformation, in terms of the original Cartesian system, can be written as

$$\tilde{x}^1 = x \qquad x = \tilde{x}$$
$$\tilde{x}^2 = y \qquad y = \tilde{x}^2$$
$$\tilde{x}^3 = \sigma(x, y, z, t) \qquad z = h(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), t$$

Should be , t) with the t inside the brackets

The functional form of $\sigma$ has been specified in a number of forms, including

$$\sigma = \theta \qquad \sigma = s(z - z_G)/(s - z_G)$$
$$\sigma = p, \qquad \sigma = (p_G - p)/(p_G - p_T),$$
$$\sigma = p/p_G \qquad \sigma = (\theta - \theta_T)/(\theta_G - \theta_T).$$
$$\sigma = \left[\frac{p - p_T}{p_G - p_T}\right]\left[\frac{p_{ref}(0) - p_T}{p_{ref}(z_G) - p_T}\right]$$

In these expressions, $p_G$, $\theta_G$, $p_T$, and $\theta_T$ refer to the pressures and potential temperatures at the bottom and top of the coordinate representation; $z_G$ and $s$ specify the terrain height and height of the top; and $p_{ref}(0)$ and $p_{ref}(z_G)$ are the pressure at sea level and at $z_G$ using a standard reference atmosphere that is the same across the model (Black 1994). The first two forms of $\sigma$ on the left are referred to as *isentropic* and *isobaric* representations, and the remaining six are *terrain-following* coordinate systems, usually called *sigma* representations. The bottom formulation in the right column for $\sigma$ is a normalized isentropic representation introduced by Branković (1981).

The innovative form of $\sigma$ at the bottom of the lefthand column is called the "Eta coordinate system" (Janjić *et al.* 1988, Janjić 1990; Mesinger and Black 1992; Black 1994; Mesinger 1996, 1997, 1998; Mesinger *et al.* 1997) and is the system used by the U.S. National Centers for Environmental Prediction (NCEP) for one of their regional models. The Eta system has the advantage of a form of sigma system that is nearly horizontal, while meeting the requirement that the system not intersect the terrain. Gallus and Klemp (2000) provide a recent comparison of model simulations of airflow over mountains using the Eta coordinate and another form of a terrain-following coordinate system. In ocean models, a coordinate system that uses density as a vertical coordinate is often used (see, e.g., Bleck and Boudra 1981). Adcroft *et al.* (1997) and Marshall *et al.* (1997) use a partial grid volume coordinate system (called "shaved cells") at their ocean bottom–ocean interface. Laprise (1992a) suggests using hydrostatic pressure as the vertical coordinate.

Correction: (Janjic 1990; Mesinger and Black 1992; Black 1994; Mesinger 1996, 1997, 1998; Mesinger et al. 1997, 1998) Reported by Fedor Mesinger, NOAA

Error reported by Jerry Davis, North Carolina State University, 10/14/02



# The Sigma-$z$ Coordinate System

Terrain-following coordinate systems that are a function of $z$ have been used extensively in regional and mesoscale models (e.g., Mahrer and Pielke 1975; Colton 1976; Blondin 1978; Yamada 1978a) in which the hydrostatic assumption has been applied and in mesoscale models in which the hydrostatic assumption has not been made (e.g., Gal-Chen and Somerville 1975a, b; Clark 1977; Pielke *et al.* 1992; Shi *et al.* 2000).

### 6.3.1 The Hydrostatic Assumption Derivation

In developing hydrostatic model equations, investigators have generally applied the chain rule *separately* in the vertical and horizontal dimensions (using the hydrostatic relation). Using the terrain-following coordinate system defined by

$$\sigma = s\frac{z - z_G}{s - z_G}, \tag{6-48}$$

for example, where $s$ is a constant and $z_G$ is a function of $x$ and $y$, application of the chain rule to the hydrostatic relation given by Eq. (4-40) yields

Should be Eq. (4-41)

$$\frac{\partial \bar{\pi}}{\partial \sigma} = -\frac{s - z_G}{s}\frac{g}{\bar{\theta}}. \tag{6-49}$$

Applying the chain rule separately to Eq. (4-41) is appropriate if the hydrostatic assumption is *exactly* satisfied. However, the invariance of the physical representation is lost if the assumption is not exact, as discussed by Dutton (1976:242), since a correct tensor transformation is required. When the horizontal scales are much larger than the vertical scales of motion, the hydrostatic relation is very closely satisfied, and such a separation of the vertical and horizontal equation may be justified. By making the hydrostatic assumption before the coordinate transformation, however, significant insight into the effect of the change of coordinates on the form of the physical invariance of the conservation relations in the transformed system cannot be evaluated. To provide such insight, it is necessary to use the methods of tensor analysis to transform coordinate systems, and then to invoke a more general form of the hydrostatic assumption. A more in-depth understanding of the coordinate transformation is then obtained.

To examine the effect of using the hydrostatic assumption in Eqs. (6-40), (6-41), and (6-42), Eq. (6-48) is defined as the generalized vertical coordinate.

The velocity vector $\vec{V}$, can be expressed as (Pielke and Cram 1989)

Should be tau_i not tau_j

$$
\begin{aligned}
\vec{V} &= \tilde{u}_i\vec{\eta}^i = \tilde{u}^i\vec{\tau}_j = u\vec{i}+v\vec{j}+w\vec{k},\\
\vec{V} &= \tilde{u}_1\vec{i}+\tilde{u}_2\vec{j}+\tilde{u}_3\left[\vec{i}\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial z_G}{\partial x}+\vec{j}\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial z_G}{\partial y}+\vec{k}\left(\frac{s}{s-z_G}\right)\right],\\
\vec{V} &= \tilde{u}^1\left[\vec{i}+\vec{k}\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1}\right]+\tilde{u}^2\left[\vec{j}+\vec{k}\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2}\right]\\
&\quad +\tilde{u}^3\vec{k}\left(\frac{s-z_G}{s}\right).
\end{aligned}
\tag{6-53}
$$

The velocities $\tilde{u}_i$ and $\tilde{u}^i$ are the covariant and contravariant components, respectively, and are given by

$$
\begin{aligned}
\tilde{u}^1 &= u,\\
\tilde{u}^2 &= v,\\
\tilde{u}^3 &= u\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial z_G}{\partial x}+v\left(\frac{\sigma-s}{s-z_G}\right)\frac{\partial z_G}{\partial y}+w\left(\frac{s}{s-z_G}\right),\\
\tilde{u}_1 &= u+\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1}w,\\
\tilde{u}_2 &= v+\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2}w,\\
\tilde{u}_3 &= \left(\frac{s-z_G}{s}\right).
\end{aligned}
$$

Therefore, the vectors in Eq. (6-53) can be rewritten in terms of the Cartesian quantities as

$$
\begin{aligned}
\vec{V} &= \left[u+\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1}w\right]\vec{i}+\left[v+\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2}w\right]\vec{j}\\
&\quad + w\left(\frac{1}{s}\right)\left[\vec{i}(\sigma-s)\frac{\partial z_G}{\partial x}+\vec{j}(\sigma-s)\frac{\partial z_G}{\partial y}+\vec{k}\right]\\
\vec{V} &= u\left[\vec{i}+\vec{k}\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^1}\right]+v\left[\vec{j}+\vec{k}\left(\frac{s-\sigma}{s}\right)\frac{\partial z_G}{\partial \tilde{x}^2}\right]\\
&\quad + s\left[u(\sigma-s)\frac{\partial z_G}{\partial x}+v(\sigma-s)\frac{\partial z_G}{\partial y}+w(s)\right]\vec{k}\left(\frac{1}{s}\right).
\end{aligned}
\tag{6-55}
$$

Figure 6-4 shows the vector $\vec{V}$ presented in the Cartesian, covariant, and contravariant forms for a two-dimensional case.

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Substituting Eq. (7-7) into Eq. (7-6) yields

$$\mathrm{R_f} = K_\theta \frac{g}{\theta_0} \frac{\partial \bar{\theta}}{\partial z} \Big/ K_m \left[ \left( \frac{\partial \bar{u}}{\partial z} \right)^2 + \left( \frac{\partial \bar{v}}{\partial z} \right)^2 \right]$$

$$- \frac{K_\theta}{K_m} \frac{g}{\theta_0} \frac{\partial \bar{\theta}}{\partial z} \Big/ \left[ \left( \frac{\partial \bar{u}}{\partial z} \right)^2 + \left( \frac{\partial \bar{v}}{\partial z} \right)^2 \right] = \frac{K_\theta}{K_m} \mathrm{Ri}, \qquad (7\text{-}8)$$

where Ri is called the *gradient Richardson number*. The sign of Ri is determined by the sign of the lapse rate of potential temperature. Thus the following conditions apply:

- $\mathrm{Ri} > 0$ corresponds to $\partial\bar{\theta}/\partial z > 0$, which indicates a stably-stratified layer.
- $\mathrm{Ri} = 0$ corresponds to $\partial\bar{\theta}/\partial z = 0$, which corresponds to neutral stratification.
- $\mathrm{Ri} < 0$ corresponds to $\partial\bar{\theta}/\partial z < 0$, which indicates an unstably stratified layer.

Theory (e.g., Dutton 1976:79) indicates that when Ri is greater than 0.25, the stable stratification sufficiently suppresses turbulence so that the flow becomes laminar, even in the presence of mean wind shear. This value of Ri is called the *critical Richardson number*.

The unstable-stratified layer itself is broken down into two regimes:

- $|\mathrm{Ri}| \le 1$, where the shear production of subgrid-scale kinetic energy is important (a regime referred to as *forced convection*).
- $|\mathrm{Ri}| > 1$, where the shear production becomes unimportant relative to the buoyant product of subgrid-scale kinetic energy (a regime called *free convection*).

The characteristic size of turbulent eddies in the atmosphere are larger during free convection than under forced convection. Brutsaert (1999) provides a recent review of boundary-layer turbulence during free convection.

should be m/s

As reported in Turner (1969), the intensity of turbulence near the ground can be estimated straightforwardly using a wind speed of 10 m, incoming solar radiation, cloud cover, and time of day. The stability classification scheme discussed by Turner forms the foundation of most air quality assessments on the mesoscale in the United States today. Unfortunately, although the dispersion estimates were developed from observations of diffusion over flat, horizontally homogeneous terrain, Gaussian plume models using these estimates are being applied for a wide range of mesoscale systems that are neither flat nor homogeneous. As reported by the American Meteorological Society in a position paper (AMS 1978), over flat, horizontally homogeneous terrain, Gaussian plume models probably give estimates of downwind plume concentrations within a

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By a Taylor series expansion,

$$\sin k\Delta x = k\Delta x - \frac{(k\Delta x)^3}{3!} + \frac{(k\Delta x)^5}{5!}$$

Thus, when $k\Delta x \ll 1$,[2] Eq. (10-4) can be written as

$$\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} \bigg/ \frac{\partial \phi}{\partial x} \sim \frac{k\Delta x}{k\Delta x} = 1.$$

Since $k = 2\pi/L$, writing $L$ in terms of the grid spacing $L = n\Delta x$, where $n$ is the number of grid points in one cycle of the cosine function, $k\Delta x \ll 1$ requires that $2\pi/n \ll 1$ or $n \gg 1$. In other words, the cosine wave must have a very long wavelength for its derivative to be represented accurately by Eq. (10-2).

In contrast, if $L = 2\Delta x$, then

$$\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} \bigg/ \frac{\partial \phi}{\partial x} = \frac{\sin \pi}{\pi} = 0,$$

so that the representation given by Eq. (10-2) fails to resolve a feature that has a wavelength of two grid increments. Examples of a longwave and a shortwave are given in Figure 10-2.

Thus the representation of the derivative of a function using values at neighboring grid points provides very poor representations of short waves relative to the grid mesh $\Delta x$, whereas longer waves are reasonably well resolved. The ability, or lack thereof, of a numerical scheme to resolve features of different wavelengths properly is a crucial consideration in the use of a numerical approximation scheme.

The linear stability of Eq. (10-1) can be evaluated using the techniques for representing waves in terms of complex variables introduced in Chapter 5.[3] As discussed there, a dependent variable $\phi$, for example, can be represented as

$$\phi(x, t) = \hat{\phi}(k, \omega)e^{i(kx+\omega t)} \qquad (10\text{-}5)$$

where $\hat{\phi}$, $k$, and $\omega$ can be complex. In a numerical model, the spatial and temporal independent variables can be written as

Should be n delta x

$$x = n\Delta z \qquad \text{and} \qquad t = \tau\Delta t,$$

so that Eq. (10-5) can also be written as

$$\phi(x, t) = \phi(n\Delta x, \tau\Delta t) = \hat{\phi}(k, \omega)e^{i(kn\Delta x+\omega\tau\Delta t)} \qquad (10\text{-}6)$$

As discussed in Chapter 5, to use the formulation given by Eq. (10-5) in a differential equation, it is necessary to linearize the equation. As written, Eq. (10-1)

Other investigators have also suggested improved techniques to conserve mass. These include Galperin and Kastrel (1998), Walcek and Aleksic (1998), and Walcek (1999a, b). The latter papers provided their Fortran code, a publication procedure that should be encouraged.

### 10.1.2 Subgrid-Scale Flux

As shown by Eq. (7-7), the subgrid-scale correlation terms can be represented as the product of an exchange coefficient and the gradient of the appropriate dependent variable. This relation can be written as, for example,

$$\frac{\partial \bar{\phi}}{\partial t} = \frac{\partial}{\partial z} K \frac{\partial \bar{\phi}}{\partial z} \simeq \frac{\phi_i^{\tau+1} - \phi_i^{\tau}}{\Delta t} = K_{i+\frac{1}{2}} \frac{\phi_{i+1}^{\tau} - \phi_i^{\tau}}{(\Delta z)^2} - K_{i-\frac{1}{2}} \frac{\phi_i^{\tau} - \phi_i^{\tau} - 1}{(\Delta z)^2}, \quad (10\text{-}25)$$

-1 should be subscript i-1 as in last term of Eq. (10-26)

where $\Delta z = z(i+1) - z(i) = z(i) - z(i-1)$ and $\phi$ represents any one of the dependent variables. This equation is often referred to as the *diffusion equation.* To study the linear stability of this scheme, the exchange coefficient is assumed to be a constant ($K_{i+1/2} = K_{i-1/2} = K$) and Eq. (10-25) is written as

$$\phi_i^{\tau+1} = \phi_i^{\tau} + K \frac{\Delta t}{(\Delta z)^2} \left( \phi_{i+1}^{\tau} - 2\phi_i^{\tau} + \phi_{i-1}^{\tau} \right). \quad (10\text{-}26)$$

The exact solution to the diffusion equation [the left side of Eq. (10-25) with $K$ equal to a constant, i.e., $\partial\bar{\phi}/\partial t = K\, \partial^2\bar{\phi}/\partial z^2$] can be determined by assuming

$$\bar{\phi} = \phi_0 e^{i(kz+\omega t)} = \phi_0 e^{-\omega_i t} e^{i(k_r z + \omega_r t)},$$

where damping in the $z$ direction is not permitted (i.e., $k_i \equiv 0$). Substituting this expression into the linearized diffusion equation and simplifying yields

$$i\omega_r - \omega_i = -Kk^2,$$

where the subscript "r" on $k$ has been eliminated to simplify the notation. Equating real and imaginary components shows that $\omega_r \equiv 0$, so the exact solution can be written as

$$\bar{\phi} = \phi_0 e^{-Kk^2 t} e^{ikz}.$$

Expressing the dependent variables as a function of frequency and wavenumber, Eq. (10-26) can be rewritten as

$$\psi^1 = 1 + \gamma(\psi_1 - 2 + \psi_{-1}) = 1 + 2\gamma(\cos k\Delta z - 1),$$