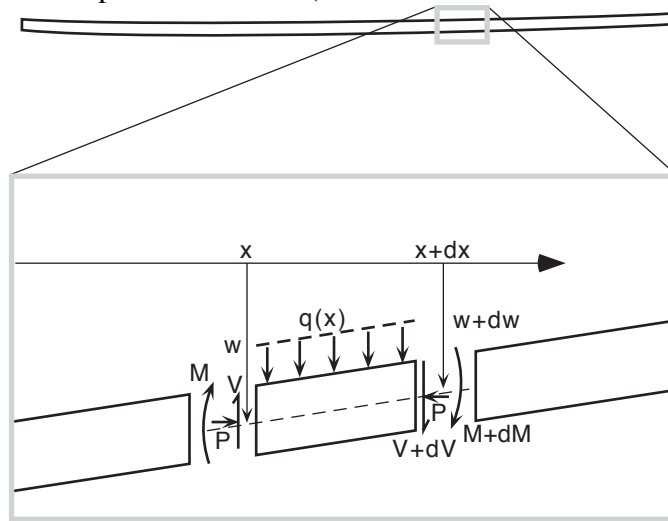


## GEOL5690 Class notes: Flexure

Flexure of the lithosphere brings together two end notions of how the Earth behaves: the rigid behavior of the Earth (after all, we don't worry about buildings or landfills causing the crust to sink down isostatically (we might worry about them sinking into peat or clays, though)) with the observation of isostatic support of the lithosphere at very long wavelengths. By bridging this gap, we find ourselves in possession of a tool capable of explaining many phenomena of interest in tectonics, from the geometry of trenches to foredeeps in front of mountain belts.

### General Equations for Flexure of Plates (Turcotte & Schubert, sec 3-9):

Consider a thin plate of elastic material. If the plate is thin enough, we may approximate the forces acting on and within the plate as follows (note arrow on  $M+dM$  backwards):



The displacement of the plate from some horizontal axis is  $w(x)$ . The forces and moments acting upon a small segment of the plate are as illustrated above, where  $q$  is a force per unit length acting on the plate,  $V$  is the shear force acting on the edges of a piece of the plate (integrated shear stresses on the edge),  $P$  is the pressure on the edges (integrated normal stresses; must balance because there is no motion of the plate to the right or left),  $M$  is the moment acting on the edges of the piece of the plate (moment of the normal stresses). If we balance forces in the vertical direction, we find

$$q(x)dx + dV = 0$$

$$\frac{dV}{dx} = -q \quad (1)$$

Balancing the forces in the horizontal ( $x$ ) direction shows that  $P$  is constant. Summing the moments acting on the segment (and noting that we can drop the second order term  $dV dx$ ) we find

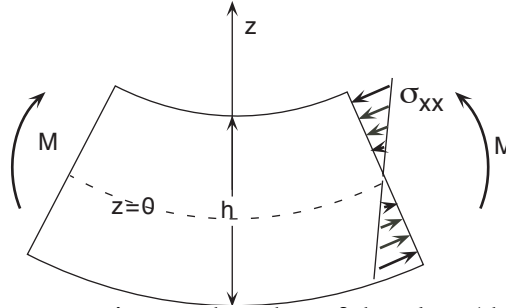
$$dM - Pdw = Vdx$$

$$\frac{dM}{dx} = V + P \frac{dw}{dx} \quad (2)$$

We drop  $V$  by combining (1) and (2) to get

$$\frac{d^2 M}{dx^2} = -q + P \frac{d^2 w}{dx^2} \quad (3)$$

If we examine  $M$  more closely, we find it is related to  $w$ :



First, consider the stresses acting on the edge of the plate (these are deviatoric stresses; we have subtracted out the mean stress  $P/h$ ). The total moment is

$$M = \int_{-h/2}^{h/2} \sigma_{xx} z dz \quad (4)$$

Now if we consider the plate in its original unbent state and compare it to the diagram above, we can see that there has been shortening towards the top and lengthening below. The dashed line is the neutral surface, which is unstrained by the bending. From Hooke's law relating stress to strain in a linear, isotropic elastic medium we have

$$\begin{aligned} \epsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \\ \epsilon_{yy} &= \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) \end{aligned} \quad (5)$$

where  $E$  is Young's modulus (of order 70 GPa for the crust) and  $\nu$  is Poisson's ratio (of order 0.25 for the crust) and  $y$  is perpendicular to  $x$  in the horizontal plane. For our purposes now, we are going to say that all variations are in our cross section, that is,  $\epsilon_{yy} = 0$ . We may then combine these equations to get

$$\sigma_{xx} = \frac{E}{(1 - \nu^2)} \epsilon_{xx} \quad (6)$$

this can be placed into (4) to get

$$M = \frac{E}{(1 - \nu^2)} \int_{-h/2}^{h/2} \epsilon_{xx} z dz \quad (7)$$

Now the strains are geometrically related to the radius of curvature of this piece of plate,  $R$ , assuming the radius of curvature remains large:

$$\epsilon_{xx} = -\frac{\Delta l}{l} = \frac{z}{R} \quad (8)$$

With a little geometry we find that the angle  $\phi$  subtended by the length  $l$  (about  $dx$ ) is

$$\phi = d\alpha = \frac{d\alpha}{dx} dx = \frac{d}{dx} \left( -\frac{dw}{dx} \right) dx = -\frac{d^2 w}{dx^2} dx \quad (9)$$

Since  $\phi = l/R$  we can find that

$$\frac{1}{R} = \frac{\phi}{l} \approx \frac{\phi}{dx} = -\frac{d^2w}{dx^2} \quad (10)$$

Substitution back into (8) gets us

$$\epsilon_{xx} = -z \frac{d^2w}{dx^2} \quad (11)$$

This then can go into (7) to get the moment:

$$\begin{aligned} M &= \frac{-E}{(1-\nu^2)} \frac{d^2w}{dx^2} \int_{-h/2}^{h/2} z^2 dz \\ &= \frac{-E}{(1-\nu^2)} \frac{d^2w}{dx^2} \left( \frac{z^3}{3} \right)_{-h/2}^{h/2} \\ &= \frac{-Eh^3}{12(1-\nu^2)} \frac{d^2w}{dx^2} \end{aligned} \quad (12)$$

This last gives us the flexural rigidity of the plate,  $D$ :

$$D \equiv \frac{Eh^3}{12(1-\nu^2)} \quad (13)$$

If we now use this in equation (3) we get

$$D \frac{d^4w}{dx^4} = q(x) - P \frac{d^2w}{dx^2} \quad (14)$$

This is now a differential equation that can be solved for the deformation of a plate in two dimensions. It is the general equation governing deflection of a thin plate without large deflections. We shall now consider how it can be applied to deformation of the lithosphere, the problem we have at hand. This is the basis for deflection of sheets of elastic materials in engineering (which may well be one of the most common applications of this piece of physics)

### Basic application to the Earth's lithosphere (Turcotte and Schubert sec. 3-13)

The very first aspect that we encounter for our purposes that is different than those typical engineering applications is that our plates are not in air. We treat the upper part of the Earth as an elastic plate overlying an inviscid substrate in the asthenosphere. That is to say that our elastic plate is over a fluid. Thus if we pushed down on part of the plate, we would not only be resisted by the forces that keep the plate from bending, as above, but also the pressure of the fluid asthenosphere on the base of the plate.

Consider lithosphere of thickness  $h$  and density  $\rho_m$  over asthenosphere of density  $\rho_a$  in isostatic equilibrium under a column of water thickness  $h_w$ . At a depth  $z$  under the lithosphere, the pressure will be the integral of the weight above, or (per unit area)  $g\rho_w h_w + g\rho_m h + g\rho_a (z - h_w - h)$ . Let us assume that this defines the pressure in the asthenosphere. Now consider part of the lithosphere that has been depressed a distance  $w$  under a load. If a material of density  $\rho_f$  fills in above the plate, we have the weight above the base of the lithosphere as being  $\rho_w g h_w + g\rho_f w + g\rho_m h$ . The force per unit area acting on the base of the lithosphere is the asthenospheric pressure at  $h_w + h + w$  minus the weight of the column above, or

$$g [\rho_w h_w + \rho_m h + \rho_a (h_w + h + w - h_w - h) - \{\rho_w h_w + \rho_f w + \rho_m h\}] = (\rho_a - \rho_f) g w \quad (15)$$

This force is directed upwards. If we are defining down as positive for deflection and force, then the net force per unit area acting on the lithosphere is

$$q(x) = q_a(x) - (\rho_a - \rho_f) g w \quad (16)$$

where  $q_a(x)$  is the load applied at the top of the lithosphere. Our equation (14) now becomes

$$D \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} + (\rho_a - \rho_f) g w = q_a(x) \quad (17)$$

If the material filling in the top has a density of crust, say  $\rho_c$ , then  $\rho_c$  can be substituted for  $\rho_f$ .

### Periodic topography (Turcotte and Schubert, sec. 3-14)

The simplest place to start is with a solution that requires little effort and yet gives us our first real insights into the scales where isostasy takes over from rigidity. Consider some region where topography is sinusoidal with  $x$  such that the elevation  $e(x) = e_0 \sin(2\pi x/\lambda)$  and does not vary with  $y$ . The load on the lithosphere,  $q_a$ , is then the variation of weight that accompanies this deflection:

$$q_a(x) = \rho_c g e_0 \sin 2\pi \frac{x}{\lambda} \quad (18)$$

where  $\rho_c$  is the density of the crust associated with the height variation. If we now assume there is no end load pressure  $P$ , we find in eqn. (17) that

$$D \frac{d^4 w}{dx^4} + (\rho_a - \rho_c) g w = \rho_c g e_0 \sin 2\pi \frac{x}{\lambda} \quad (19)$$

Note that we use a fill density of  $\rho_c$ , which means that our applied load is added on a level upper surface and so is the final (observed) topography of the system (see addendum at end of these notes). Given the periodic nature of the load, we would expect that the deflection of the lithosphere would also be periodic. Let us guess a solution of the form

$$w(x) = w_0 \sin 2\pi \frac{x}{\lambda} \quad (20)$$

Putting this into equation 19 produces

$$D w_0 \left( \frac{2\pi}{\lambda} \right)^4 \sin 2\pi \frac{x}{\lambda} + (\rho_a - \rho_c) g w_0 \sin 2\pi \frac{x}{\lambda} = \rho_c g e_0 \sin 2\pi \frac{x}{\lambda}$$

$$w_0 \left[ D \left( \frac{2\pi}{\lambda} \right)^4 + (\rho_a - \rho_c) g \right] = \rho_c g e_0 \quad (21)$$

$$w_0 = \frac{e_0}{\frac{D}{g \rho_c} \left( \frac{2\pi}{\lambda} \right)^4 + \frac{\rho_a}{\rho_c} - 1}$$

We notice two endmembers: one where the wavelength is very small, one where it is very large.

When  $\lambda$  is small (much smaller than  $2\pi \left( \frac{D}{g \rho_c} \right)^{1/4}$ ), the first term in the denominator dominates

and  $w_0$  approaches zero. This is the familiar case of the rigid earth not yielding beneath a landfill; the wavelength of the landfill turns out to be much smaller than the flexural strength of the lithosphere. In the other extreme, as  $\lambda$  goes to infinity (at least much greater than

$2\pi \left( \frac{D}{g\rho_c} \right)^{1/4}$ ) then the first term drops out and we are left with  $w_0 = e_0 \rho_c / (\rho_a - \rho_c)$ , which is the equation for local, Airy isostatic support of mountains. (For references, you might note that the factor  $\rho_c / (\rho_a - \rho_c)$  is how much more the Moho is perturbed for a given topographic signal; it varies from about 4 to 8, so 1 km high mountains should have 4-8 km deeper Mohos than adjacent lowlands). For values of the wavelength in between, the topography is partially compensated; this is illustrated in Figure 3-26 of Turcotte and Schubert. As an aside, this "partial compensation" was used in the Bond and Kominz backstripping paper to allow for shorter wavelength loads of the rifted margin to only partially deflect the Moho.

The sinusoidal solution can be the basis for a Fourier expansion of the response to a load and is one potential strategy for solving more complex loads. The wavelength where support for topography transitions from rigid to isostatic depends on the flexural rigidity  $D$  of the plate, and determining this transition in the basis for so-called admittance analysis of gravity (a proxy for the deflection  $w$  of the Moho) against topography that has been very successful in revealing the rigidity of oceanic lithosphere. For the cases of greatest interest here, there are simpler solutions.

### Linear load on unbroken lithosphere (sec. 3-16)

Things are somewhat more complicated when we make our load finite. Let us start with a simple load isolated along a line; let the magnitude of this force be  $V_0$  at  $x = 0$ . Elsewhere the load is zero. We once again let the end load be negligible and wish to solve (17) everywhere but  $x = 0$  and thus face

$$D \frac{d^4 w}{dx^4} + (\rho_a - \rho_w) g w = 0 \quad (22)$$

There is a general solution to this equation:

$$w = e^{x/\alpha} \left( c_1 \cos \frac{x}{\alpha} + c_2 \sin \frac{x}{\alpha} \right) + e^{-x/\alpha} \left( c_3 \cos \frac{x}{\alpha} + c_4 \sin \frac{x}{\alpha} \right) \quad (23)$$

where the flexural parameter  $\alpha$  is defined as

$$\alpha = \left( \frac{4D}{g(\rho_a - \rho_w)} \right)^{1/4} \quad (24)$$

There is a natural symmetry about  $x = 0$ , so we can limit our efforts for  $x > 0$ . For distances far from the load, we expect the deflection to go to 0; this causes the positive exponent terms to be dropped ( $c_1$  and  $c_2 = 0$ ). Because our plate is continuous and symmetric, the slope  $= dw/dx$  must be zero at  $x = 0$ . This in turn forces  $c_3 = c_4$ .

The last constant is related to the magnitude of our load  $V_0$ . If we return to equation (2) and let  $P = 0$ , we find that  $V = dM/dx$ , which, from equation (12), is

$$V = \frac{-Eh^3}{12(1-\nu^2)} \frac{d^3 w}{dx^3} = -D \frac{d^3 w}{dx^3} \quad (25)$$

Since our load is pointing down (opposite the sense from the top figure), and since we are solving the half of this for  $x > 0$ , we find that

$$\frac{V_0}{2} = \frac{4Dc_3}{\alpha^3} \quad (26)$$

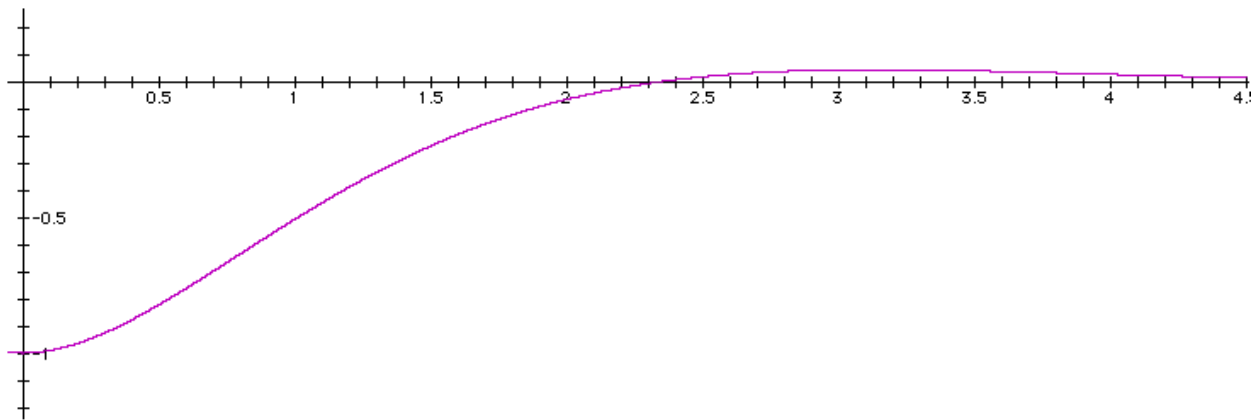
We can now return to (23) and find our solution to be

$$w = w_0 e^{-x/\alpha} \left( \cos \frac{x}{\alpha} + \sin \frac{x}{\alpha} \right) \quad (27)$$

where our maximum deflection  $w_0$  is at  $x = 0$  and is

$$w_0 = \frac{V_0 \alpha^3}{8D} \quad (28)$$

There are several characteristics of this solution worth examination. First is to note that  $\alpha$  does not depend on  $V_0$ . Thus an increase in the load does not alter the shape of the solution, only its amplitude. This is particularly noteworthy because the solution varies about  $w = 0$ :



Here the vertical axis is  $w/w_0$  and the horizontal is  $x/\alpha$ . The crossing of  $w = 0$  at  $x = 0.75\pi\alpha$  separates the depression from the forebulge. The position of this crossing is unaffected by changing  $w_0$  (or  $V_0$ ); thus the position of the forebulge is only affected by the position of the load and  $\alpha$ , which depends on the physical characteristics of the lithosphere. Note that the forebulge is in general much smaller than the depression itself, only reaching a maximum amplitude just under 5% of the main depression. (Also note that  $\alpha$  depends on the density contrast of the infilling material and the asthenosphere--what happens when a sediment starved basin is filled?).

(Aside: the total deflection integrated to infinity can be of some use, as in some instances we might be able to obtain a lower bound on this integral. So integrating (27) yields

$$\int w(x) dx = -\alpha w_0 e^{-x/\alpha} \cos \frac{x}{\alpha} + \text{constant}$$

which, if evaluated from 0 to infinity is simply  $-\alpha w_0$ . If we double this (to collect the symmetrical contribution) and insert (28) and (24) we find that the integrated deflection is

$$\int w(x) dx = \frac{V_0 \alpha^4}{4D} = \frac{V_0}{g(\rho_a - \rho_w)}$$

$$V_0 = g(\rho_a - \rho_w) \int w(x) dx$$

Basically, this says that the total load has to equal the total weight of displaced material, which is simply restating Archimedes's Principle. This should not be a surprise and can be used as a check on the validity of more complex solutions.)

### Flexure of a broken plate (sec. 3-17; also see sec. 12.1, 12.2 for Matlab implementations)

The final instance we are interested in is if the plate is broken. Strictly speaking this simply doesn't happen--there is never an open edge to a lithospheric plate. But once again this kind of approximation gives us insight to situations where the strength of the lithosphere might vary dramatically where the load is applied.

In this instance we no longer require that the slope of the plate be flat at  $x = 0$ ; we replace this with the requirement that the moment be zero on the end of the plate (there is no source of moment in our initial analysis). From (12) we then require that the second derivative of  $w$  with  $x$  is 0 at  $x = 0$ . Returning to eqn (23) we now find that  $c_1$ ,  $c_2$ , and  $c_4$  must be zero. Repeating the exercise that went into equations (25) and (26), we find that

$$\frac{V_0}{2} = \frac{2Dc_3}{\alpha^3} \quad (29)$$

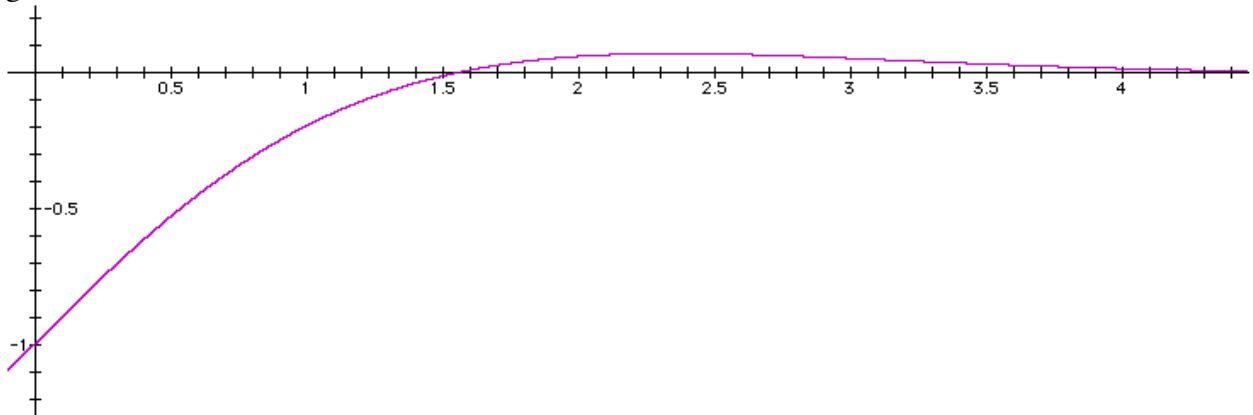
Plugging back in, we find

$$w = w_0 e^{-x/\alpha} \cos \frac{x}{\alpha} \quad (30)$$

where

$$w_0 = \frac{V_0 \alpha^3}{4D} \quad (31)$$

Plotting on the same horizontal scale as above (and with  $w/w_0$  with the new  $w_0$ ), we see several things.



As before, the zero crossings and forebulge positions rely on  $\alpha$  exclusively. The zero crossing has moved in considerably closer to the load for a given  $\alpha$ , as has the forebulge. The amplitude of the forebulge is now nearly 7% of the maximum depression (which itself is twice the amplitude as for the unbroken plate). When trying to fit a given deflection profile using a broken plate will require about a 50% thicker effective elastic plate thickness than fitting the same profile with a continuous plate. The largest difference in the shapes of the plots is near the load, which is where the assumption of a load only at  $x = 0$  breaks down.

Additional variations are possible where a moment is placed on the end of the elastic plate; this is covered in detail in Turcotte and Schubert; it is most applicable when a large subsurface load (usually a subducted plate) is present. Use of this formulation predicts the observed topography at ocean trenches quite well, as well as the deflection of (unfaulted) continental sedimentary basins, as we shall discuss further.

### Addendum: Periodic Loading vs. Periodic Topography

If our derivation of the periodic response to topographic loading made sense above, then you may want to ignore what follows. This derivation actually treats the load term as what has been piled onto an originally uniform plate; you might prefer it in instances where you are wondering of the response related to, say, an isopach map of sediments. The basic relation for thickening crust in isostatic equilibrium by some amount  $d$  is that it will increase the elevation  $e$  relative to unthickened crust by an amount  $d(\rho_a - \rho_c)/\rho_a$ . Alternatively, we can express this as the ratio of the deflection of the Moho  $w$  to the elevation  $e$ . Noting that  $d = w + e$  in all cases, we find that  $e = w(\rho_a - \rho_c)/\rho_c$ .

Let us place on a uniform plate a specified thickness of rock,  $d(x) = d_0 \sin(2\pi x/\lambda)$  and see how the lithosphere reacts (we will use  $w'$  and  $e'$  for our specific solutions just to separate them from the  $w$  and  $e$  we derived above). In this case the material filling the voids will have the density of air ( $\rho_f = 0$  for all intents and purposes). Thus the final topography  $e(x)$  will be  $d(x)$  minus the deflection  $w(x)$  (right hand diagram below). To demonstrate, consider that eqn. 18 becomes:

$$q_a(x) = \rho_c g d_0 \sin 2\pi \frac{x}{\lambda} \quad (\text{A1})$$

where  $\rho_c$  is the density of the crust associated with the height variation. If we now assume there is no end load pressure  $P$ , and set  $\rho_f$  to 0, we find in eqn. (17) that

$$D \frac{d^4 w}{dx^4} + \rho_a g w = \rho_c g e_0 \sin 2\pi \frac{x}{\lambda} \quad (\text{A2})$$

Again we try a solution like

$$w(x) = w'_0 \sin 2\pi \frac{x}{\lambda} \quad (\text{A3})$$

Putting this into equation A1 produces

$$\begin{aligned} D w'_0 \left( \frac{2\pi}{\lambda} \right)^4 \sin 2\pi \frac{x}{\lambda} + \rho_a g w'_0 \sin 2\pi \frac{x}{\lambda} &= \rho_c g d_0 \sin 2\pi \frac{x}{\lambda} \\ w'_0 \left[ D \left( \frac{2\pi}{\lambda} \right)^4 + \rho_a g \right] &= \rho_c g d_0 \\ w'_0 &= \frac{d_0}{\frac{D}{g \rho_c} \left( \frac{2\pi}{\lambda} \right)^4 + \frac{\rho_a}{\rho_c}} \end{aligned} \quad (\text{A4})$$

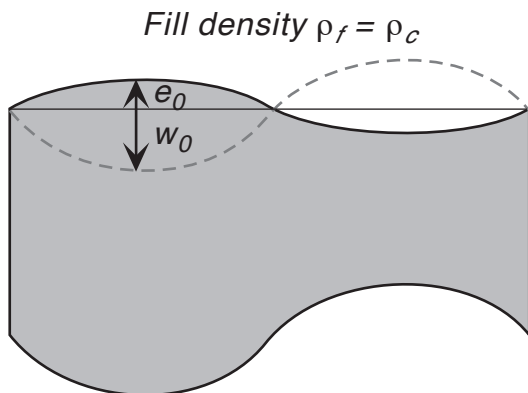
Since the final topography  $e'(x)$  will be  $d(x)$  minus the deflection  $w'(x)$ , we get:

$$e'_0 = d_0 - w'_0 = d_0 - \frac{d_0}{\frac{D}{g \rho_c} \left( \frac{2\pi}{\lambda} \right)^4 + \frac{\rho_a}{\rho_c}} \quad (\text{A4})$$

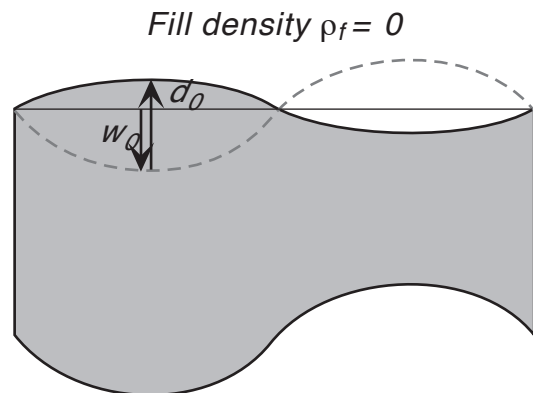
Whereas our original solution showed how periodic topography of different wavelengths will require different deflections of the Moho, increasing towards the isostatic limit (long wavelengths), this derivation complements this by showing how the topography from a periodic load of fixed amplitude will decrease toward the isostatic limit. This is pretty intuitive: a short

load (like a building) will not deflect the lithosphere and so the local topography will be the building height; a broad load like a wide mountain belt will sink down, and the topography will be a lot less than the pile of rock put on the plate. In the isostatic limit (as  $\lambda$  goes to  $\infty$ ), this reduces to  $e'_0 = d_0(\rho_a - \rho_c)/\rho_a$  as it should.

In our original solution above, we used the form of eqn (17) that had deflections filled with rock of density  $\rho_c$ . Thus our periodic load representing rock of thickness  $e_0 \sin(2\pi x/\lambda)$  and density  $\rho_c$  is in fact the *final* topography and not the total load, which is instead the final topography plus the filling in of the deflection  $w$ . This is evident from the lefthand figure below, where the crust is shaded gray,  $e$  is the amplitude of the load, and we can see that the holes generated by flexure have been filled with material with crustal density (and bulges have been removed). (An alternative way of viewing this is that the load is the only deviation from a planar top surface that results from the infilling with density  $\rho_c$ ). Using the solution we obtained from (21) as  $\lambda$  became large, we see that the topography  $e$  does have the correct relation to the deflection of the Moho.



Moho deflection =  $w$   
 If in isostatic equilibrium,  
 $w_0 = e_0 \rho_c / (\rho_a - \rho_c)$  or  
 topography =  $e_0 = w_0 (\rho_a - \rho_c) / \rho_c$



Moho deflection =  $w$   
 If in isostatic equilibrium,  
 $w_0 = d_0 \rho_c / (\rho_a - 0) = d_0 \rho_c / \rho_a$   
 but topography  $e_0 = d_0 - w_0$   
 $= d_0 (\rho_a - \rho_c) / \rho_a = w_0 (\rho_a - \rho_c) / \rho_c$

Although these two formulations yield the same results (as well they should!), each would probably be used in different circumstances. The first (fill density = density of the crust) would be used when we know the surface topography and wish to know the deflections compatible with the assumption that such topography was built by putting material on the surface. The second (fill density of air or perhaps water) would be useful when we have a known load, perhaps from an isopach map, and wish to know the resulting topography.