Dispersion relation for acoustic-gravity waves including the Coriolis terms

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We first start with [Jones(2006), equation (10)], which neglects rate of strain in the dispersion relation as a special case of the general dispersion relation given by [Jones(2006), equation (5)]

$$\begin{aligned}(\mathbf{k}^2+\mathbf{k}_A^2)(N^2-\omega^2)+\mathbf{k}\cdot\mathbf{S}\cdot\mathbf{k}+\mathbf{k}_A\cdot\mathbf{S}\cdot\mathbf{k}_A+\mathbf{A}\cdot\mathbf{k}\\ +1/C^2(\omega^4-4\omega^2\tilde{\mathbf{\Omega}}^2+B^2/2-2i\omega\tilde{\mathbf{\Omega}}\cdot\mathbf{B})=0\end{aligned} \quad , \tag{1}$$

where N is the Brunt-Väisälä frequency, $\omega = \sigma - \mathbf{k}\cdot\mathbf{U}$ is the intrinsic frequency, σ is the wave frequency, $\mathbf{U}$ is the background fluid velocity, $\mathbf{k}$ is the wavenumber, $\mathbf{B}$ is the baroclinic vector, $\tilde{\mathbf{\Omega}} = \mathbf{\Omega} + \zeta/4$, where $\zeta = \nabla\times\mathbf{U}$ is vorticity, $\mathbf{\Omega}$ is the Earth's angular velocity, C is sound speed, $\mathbf{k}_A \equiv \nabla\rho/(2\rho)$, where ρ is density, $\mathbf{S}$ is the symmetric matrix defined by

$$S_{\alpha\beta} = \underbrace{-\frac{1}{2\rho}\left(\frac{\partial\tilde{\rho}_{pot}}{\partial x_\alpha}\tilde{g}_\beta+\frac{\partial\tilde{\rho}_{pot}}{\partial x_\beta}\tilde{g}_\alpha\right)}_{1}\underbrace{+4\tilde{\Omega}_\alpha\tilde{\Omega}_\beta}_{2}\underbrace{+\frac{i}{\omega}\left(\tilde{\Omega}_\alpha B_\beta+\tilde{\Omega}_\beta B_\alpha\right)}_{3} \ , \tag{2}$$

$\tilde{\mathbf{g}} \equiv \nabla p/\rho = \mathbf{g} - D\mathbf{U}/Dt - 2\mathbf{\Omega}\times\mathbf{U}$ is the effective vector acceleration due to gravity [including (minus) the acceleration of the background flow], $\tilde{\rho}_{pot}$ is local potential density, defined by $\nabla\tilde{\rho}_{pot} = \nabla\rho - \nabla p/C^2$ [Jones(2005), Jones(2008a)], p is pressure,

$$\mathbf{A} = (4\omega\tilde{\mathbf{\Omega}} + i\mathbf{B})\times\mathbf{\Gamma} + 2\mathbf{k}_A\cdot\tilde{\mathbf{\Omega}}\mathbf{B}/\omega \ , \tag{3}$$

and $\mathbf{\Gamma} = \mathbf{k}_A - \tilde{\mathbf{g}}/C^2$ is the vector generalization [Jones(2001)] of Eckart's coefficient [Gossard and Hooke(1975), p. 90].

Term 2 in (2) is a Coriolis term, which we will be keeping. Term 3 in (2) is a baroclinic and Coriolis term, which we shall be neglecting when we neglect baroclinic terms.

Neglecting the baroclinic terms in (1), (2), and (3) gives

$$(\mathbf{k}^2+\mathbf{k}_A^2)(N^2-\omega^2)+\mathbf{k}\cdot\mathbf{S}\cdot\mathbf{k}+\mathbf{k}_A\cdot\mathbf{S}\cdot\mathbf{k}_A+\mathbf{A}\cdot\mathbf{k}+1/C^2(\omega^4-4\omega^2\tilde{\mathbf{\Omega}}^2)=0 \ , \tag{4}$$

where $\mathbf{S}$ is the symmetric matrix defined by

$$S_{\alpha\beta} = \underbrace{-\frac{1}{2\rho}\left(\frac{\partial\tilde{\rho}_{pot}}{\partial x_\alpha}\tilde{g}_\beta + \frac{\partial\tilde{\rho}_{pot}}{\partial x_\beta}\tilde{g}_\alpha\right)}_{1} \underbrace{+4\tilde{\Omega}_\alpha\tilde{\Omega}_\beta}_{2} \ , \tag{5}$$

and

$$\mathbf{A} = 4\omega\tilde{\mathbf{\Omega}} \times \mathbf{\Gamma} \ . \tag{6}$$

Substituting (5) and (6) into (4) gives

$$\begin{aligned} &(\mathbf{k}^2 + \mathbf{k}_A^2)(N^2 - \omega^2) - \tfrac{1}{\rho}\mathbf{k}\cdot\nabla\tilde{\rho}_{pot}\,\tilde{\mathbf{g}}\cdot\mathbf{k} - \tfrac{1}{\rho}\mathbf{k}_A\cdot\nabla\tilde{\rho}_{pot}\,\tilde{\mathbf{g}}\cdot\mathbf{k}_A \\ &+4(\mathbf{k}\cdot\tilde{\mathbf{\Omega}})^2 + 4(\mathbf{k}_A\cdot\tilde{\mathbf{\Omega}})^2 + 4\omega\tilde{\mathbf{\Omega}}\times\mathbf{\Gamma}\cdot\mathbf{k} + 1/C^2(\omega^4 - 4\omega^2\tilde{\mathbf{\Omega}}^2) = 0 \ . \end{aligned} \tag{7}$$

Equation (7) can be written

$$\begin{aligned} &(\mathbf{k}^2 + \mathbf{k}_A^2)(N^2 - \omega^2) - \left(k_z^2 + k_A^2\right) N^2 \\ &+4(\mathbf{k}\cdot\tilde{\mathbf{\Omega}})^2 + 4(\mathbf{k}_A\cdot\tilde{\mathbf{\Omega}})^2 + 4\omega\tilde{\mathbf{\Omega}}\times\mathbf{\Gamma}\cdot\mathbf{k} + 1/C^2(\omega^4 - 4\omega^2\tilde{\mathbf{\Omega}}^2) = 0 \ , \end{aligned} \tag{8}$$

where k_z is the component of $\mathbf{k}$ in the direction of $\tilde{\mathbf{g}}$.

We can compare (8) with the usual barotropic approximation [Jones(2006), equation (1)]:

$$(k_x^2 + k_y^2)(N^2 - \omega^2) - (\omega^2 - 4\Omega_z^2)(k_z^2 + \mathbf{k}_A^2 - \frac{\omega^2}{C^2}) = 0, \tag{9}$$

where N is the Brunt-Väisälä frequency, $\omega = \sigma - \mathbf{k}\cdot\mathbf{U}$ is the intrinsic frequency, σ is the wave frequency, $\mathbf{U}$ is the background fluid velocity, $\mathbf{k}$ is the wavenumber, k_z is its vertical component, k_x and k_y are its horizontal components, Ω_z is the vertical component of the Earth's angular velocity, C is sound speed, and $\mathbf{k}_A \equiv \nabla\rho/(2\rho)$, where ρ is density.

The two dispersion relations agree when we neglect the Coriolis term (including vorticity), but the Coriolis terms differ some. However, the main Coriolis term, the second Coriolis term in (8), agrees with the corresponding term in (9). The Coriolis terms are important only for very low frequency.

It would probably be useful, when adding the Coriolis terms to the ray tracing program, to be able to use both versions, so we can decide under what conditions the various terms are important.

References

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